Abel's theorem revisited through new developments in residue theory and effectivity *

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1 How to multiply a meromorphic form with the integration current attached to an analytic cycle

Let V be a pure $n-m>0$-dimensional closed analytic set in some open set U of $\mathbb{C}^n$ ($\mathbb{C}^n$ may as well be replaced by any n-dimensional complex manifold) ; one can attach to the *geometric* set V the *geometric integration current* on V, that is the unique ∂ and $\overline{\partial}$- closed positive current (with type (m,m)) such that, for any $(n-m,n-m)$-test form in U, which support does not intersect the singular set V_{sing} of V, one has :

$$\langle [V]\,,\varphi\rangle = \int_{V_{\text{sing}}} \varphi\,.$$

A meromorphic $(q,0)$ form ($q\leq n-m$) on V is by definition an holomorphic form on some $n-m$-dimensional complex manifold $V\setminus W$ (where W is an hypersurface of V which contains V_{sing}, which implies that $V\setminus W$ is a submanifold of U, where the concept of holomorphicity makes sense) which

*Talk given at the conference "Effectivity problems : algebraic and analytic methods, IV", Diamante, Calabria, June 2002 is the restriction to $V \setminus W$ of a meromorphic $(q, 0)$-form in a neighborhood of V in the ambiant manifold U. Such a notion is a robust one since all natural definitions of meromorphicity lead essentially to the same one (see for example theorem 1 in [16]).

Any coefficient distribution (in fact positive measure) of the integration current is regular holonomic, in the sense proposed by J.E. Björk [5] ; we will not enter here in the details of this definition, but just mention what will be important for us later, namely that given any point z_0 in V, one can find, for any distribution-coefficient u and any holomorphic function g in a neighborhood of z_0 which is not identically zero on any irreducible branch of V, a functional equation

$$\mathcal{Q}_{z_0,u,g}(\zeta\,,\partial_\zeta)[g^{\lambda+1}\otimes u] = b_{z_0,u,g}(\lambda)\,[g^\lambda \otimes u]$$

where $b \in \mathbb{Q}[\lambda]$. In fact, one can show that, if ω is a meromorphic form in a neighborhood of V whose polar set Y intersects V along some hypersurface W of V, one can naturally define $\omega \wedge [V]$ in some (sufficiently small) neighborhood V_{z_0} of some arbitrary point z_0 in V ; assume that h_{z_0} is an holomorphic function which vanishes on $Y \cap V$ in such a neighborhood ; then, one can define the current $\omega \wedge [V]$ in V_{z_0} as the value at $\lambda = 0$ of the $(m+q, m)$-current-valued function

$$\lambda \to |h_{z_0}|^{2\lambda}\omega \wedge [V]$$

(such a function happens to have all its poles in $\{\mathrm{Re}\,\lambda \leq -\epsilon\}$ for $\epsilon = \epsilon_{z_0} > 0$ small enough). The fact that such a definition does not depend on the choice of h_{z_0} shows that it induces a global construction for $\omega \wedge [V]$. Such a current is a principal value type Coleff-Herrera current [7]. Its $\overline{\partial}$ is a $(m+q, m+1)$ current, which is defined (in some open set ϖ where $\omega = \tilde{\omega}/g$, where $\tilde{\omega}$ is a $(q, 0)$ holomorphic form) as the restricted Coleff-Herrera residual current :

$$\overline{\partial}\left[\frac{1}{g}\right] \wedge \tilde{\omega} \wedge [V] = \mathrm{Res}\,\left[\begin{matrix}(\cdot) \wedge \tilde{\omega} \\ g\end{matrix}\right] = \mathrm{Res}\,\left[\begin{matrix}(\cdot) \wedge a\,\tilde{\omega} \\ ag\end{matrix}\right]$$

(the last equality which is valid for any holomorphic function a in ϖ is known as a particular case of the transformation law in restricted residue calculus). Any $(q, 0)$ meromorphic form ω on V is called *holomorphic* (in Barlet's sense [1]) if and only if $\overline{\partial}(\omega \wedge [V]) \equiv 0$. This notion differs from the notion of holomorphicity introduced by P.Griffiths in [13] (for any resolution of singularities

$\pi : \widetilde{V} \to V$, the *pull-back* $\pi^*[\omega]$ is a $(q,0)$-differential form with holomorphic coefficients on $\widetilde{W}$) ; in fact, Griffith's condition implies homolorphicity in the Barlet's sense (for a comparison of those various non equivalent criteria for holomorphicity of differential forms, see for example [16], section 2, proposition 1) ; note also that such criteria should be compared to criteria for the regularity of differential forms introduced in an algebraic setting, for example as in the work of Kunz and Waldi (up to our knowledge, such comparison has still not been clarified).

In order to deal with integration along analytic cycles instead of integration on geometric sets, we will also have to explain how to "multiply" a meromorphic form on V with any residual current of the Bochner-Martinelli type which is supported by V. Let $f_1, ..., f_M$ be holomorphic functions in some open neighborhood of V, such that $V := \{\zeta \in U \, ; \, f_1(\zeta) = \cdots = f_M(\zeta) = 0\}$, $\operatorname{codim} V = m$. For any ordered subset $\mathcal{I} := \{i_1, ..., i_k \, ; \, 1 \leq i_1 < \cdots < i_k \leq M\}$, $k = m, ..., \min(n, M)$, one can define a (q,k)-current

$$\operatorname{Res}\begin{bmatrix} (\cdot) \wedge \omega \\ \\ f_{i_1}, ..., f_{i_k} \\ \\ f_1, ..., f_M \end{bmatrix}$$

as follows : for any neighborhood ϖ of some point in V such that $\omega = \tilde{\omega}/g$ in ϖ,

$$\operatorname{Res}\begin{bmatrix} (\cdot) \wedge \omega \\ \\ f_{i_1}, ..., f_{i_k} \\ \\ f_1, ..., f_M \end{bmatrix} := \frac{(-1)^{k(k-1)/2}(k-1)!}{(2i\pi)^k} \left[\mu \|f\|^{2(\mu-k-1)} \frac{|g|^{2\lambda}}{g} \, \tilde{\omega} \wedge \overline{\partial} \|f\|^2 \wedge \Omega_{\mathcal{I}}[f] \right]_{\lambda=\mu=0}$$

where

$$\|f\|^2 := |f_1|^2 + \cdots + |f_M|^2 \quad , \quad \Omega_{\mathcal{I}}[f] := \sum_{l=1}^{k} (-1)^{l-1} \overline{f_{i_l}} \bigwedge_{\substack{j=1 \\ j \neq l}}^{k} \overline{df_{i_l}} \, ,$$

and $\left[\cdots \right]_{\lambda=\mu=0}$ means take the meromorphic continuation in (λ, μ), which happens to be holomorphic at $(0,0)$, then take its value at $(0,0)$; one can show that the definition does not depend on the choice of a denominator for ω. When $(f_1, ..., f_M)$ are such that they define V as some irreducible analytic set, it follows from a result by M. Méo [18] that one has the following equality between $(m+q, m)$ currents :

$$\omega \wedge [V] = \frac{1}{\nu(f)} \sum_{\substack{\mathcal{I} \subset \{1 ..., M\} \\ \# \mathcal{I} = m}} \text{Res} \begin{bmatrix} (\cdot) \wedge \omega \wedge df_{i_1} \wedge \cdots \wedge df_{i_m} \\ \\ f_{i_1}, ..., f_{i_m} \\ \\ f_1, ..., f_M \end{bmatrix},$$

where $\nu(f)$ denotes the Hilbert-Samuel multiplicity of $(f_1, ..., f_M)$ at a generic point on V.

In the general case, let $(f_1, ..., f_M)$ be holomorphic functions in U, defining a purely $n-m$-dimensional analytic closed subvariety V with a finite number of irreducible components $V_1, ..., V_s$; let $\nu_\sigma(f)$, $\sigma = 1, ..., s$, be the Hilbert-Samuel multiplicity of $(f_1, ..., f_M)$ at a generic smooth point x_σ on V_σ (that is the Hilbert-Samuel multiplicity of $(f_1, ..., f_M, L_1(x-x_\sigma), ..., L_{n-m}(x-x_\sigma))$ in $\mathcal{O}_{x_\sigma}$, $L_1, ..., L_{n-m}$ being $n-m$ generic linear forms) ; it follows from [3] that if $\tilde{f}_1, ..., \tilde{f}_{n+M}$ are defined as :

$$\begin{aligned} \tilde{f}_j &= \sum_{k=1}^{M} \lambda_{jk} f_k \,, \; j = 1, ..., m \\ \tilde{f}_{m+j} &= f_j^{m \max \nu_\sigma(f)} \,, \; j = 1, ..., M \end{aligned}$$

(for generic λ_{jk}'s) then, one has, for any meromorphic form ω on V,

$$\omega \wedge \left(\sum_{\sigma=1}^{s} \nu_\sigma(f) [V_\sigma] \right) = \sum_{\substack{\mathcal{I} \subset \{1 ..., M+m\} \\ \# \mathcal{I} = m}} \text{Res} \begin{bmatrix} (\cdot) \wedge \omega \wedge d\tilde{f}_{i_1} \wedge \cdots \wedge d\tilde{f}_{i_m} \\ \\ \tilde{f}_{i_1}, ..., \tilde{f}_{i_m} \\ \\ \tilde{f}_1, ..., \tilde{f}_{M+n} \end{bmatrix}.$$

2 Playing with the incidence variety to define the trace of a cycle

Let U be a $m < n$-concave open subset in $\mathbb{P}^n(\mathbb{C})$, that is for any point in U, one can find a m-plane ξ which contains x and lies completely in U.

Then, one can construct a dual open set U_m^* in the grassmanian $G(m,n)$ as

$$\xi \in U_m^* \Longleftrightarrow \xi \subset U\,.$$

The set of points

$$\{(x,\xi)\in U\times U_m^*\,;\, x\in\xi\}$$

is a submanifold in $U\times U_m^*$, called the *m-incidence variety* $\mathrm{Icd}_m(U)$; one has two projections $\pi_1 \ : \mathrm{Icd}_m(U)\to U$ and $\pi_2 \ : \mathrm{Icd}_m(U)\to U_m^*$; note that π_2 is a proper map, so that π_2 allows the push-forward of currents.

Let now $\mathcal{D}_1,...,\mathcal{D}_M$, M effective Cartier divisors in $\mathbb{P}^n(\mathbb{C})$ such that the intersection of the supports of the $\mathcal{D}_j$, $j=1,...,M$, define a-$n-m$-pure dimensional closed subvariety V in U ; for any irreducible component V_σ in V, let $\nu_\sigma(\mathcal{D})$ the Hilbert-Samuel multiplicity of the ideal $(s_1,...,s_M)$ generated by local sections of the $\mathcal{D}_j$ at a generic smooth point in V_σ.

Following the constructions developped in [13] and [16], a natural definition for the trace of ω on the $n-m$-analytic cycle

$$C(\mathcal{D}) := \sum_\sigma \nu_\sigma(\mathcal{D})\,V_\sigma$$

would be

$$\mathrm{Tr}_{C(\mathcal{D})}[\omega] := \sum_\sigma \nu_\sigma(\mathcal{D})\,\mathrm{Tr}_{V_\sigma}[\omega]\,,$$

where

$$\mathrm{Tr}_{V_\sigma}[\omega] := (\pi_2)_*\Big(\pi_1^*(\omega)\wedge\big[\pi_1^{-1}(V_\sigma)\big]\Big)\,.$$

One way (inspired by [2, 3]) to realize such a trace in terms of homogeneous sections $P_1,...,P_M$ for the line bundles $[\mathcal{D}_j]$ (such that $\deg P_1 = D_1 \geq \deg P_2 = D_2 \geq ...$), is to introduce (in homogeneous coordinates)

$$\begin{aligned}
\tilde{P}_j &= \sum_{k=1}^M \lambda_{jk}\Big(\sum_{l=0}^n \lambda_{0l} z_l\Big)^{D_1-D_k} P_k\,,\ j=1,...,m\,,\\
\tilde{P}_{m+j} &= P_j^{m\max\nu_\sigma(\mathcal{D})}\,,\ j=1,...,M
\end{aligned}$$

(where coefficients λ_{jk} are taken generic), then to set

$$\begin{aligned}
\Phi &:= \sum_{j=1}^{m+M}\frac{|\widetilde{P}_j|^2}{\|z\|^{2\deg\widetilde{P}_j}}\\
\Psi &:= \sum_{1\leq j_1<\cdots<j_{m-1}\leq m+M}\ \bigwedge_{l=1}^{m-1}\overline{\partial}\left[\frac{\overline{\widetilde{P}_{j_l}}}{\|z\|^{\deg\widetilde{P}_{j_l}}}\right]\wedge\partial\left[\frac{\widetilde{P}_{j_l}}{\|z\|^{\deg\widetilde{P}_{j_l}}}\right]
\end{aligned}$$

and to express $\mathrm{Tr}_{C(\mathcal{D})}[\omega]$ as

$$\mathrm{Tr}_{C(\mathcal{D})}[\omega] = \frac{(m-1)!}{(2i\pi)^m}(\pi_2)_* \left(\left[\pi_1^* \left(\Phi^{2(\mu-m-1)} \omega \wedge \overline{\partial}\Phi \wedge \partial\Phi \wedge \Psi \right) \right]_{\mu=0} \right) .$$

3 Closed formulaes in the case $U = \mathrm{P}^n(\mathbb{C})$ and multivariate residue calculus

Ideas developped in this section (which are more in the spirit of computational geometry) are inspired by P. Griffiths's paper [13].

Let us study the particular case where $U = \mathbb{P}^n(\mathbb{C})$ and $\mathcal{D}_1, ..., \mathcal{D}_m$ correspond to divisors defining a complete intersection in $\mathbb{P}^n(\mathbb{C})$, such that none of the supports of any component of the $\mathcal{D}_j$ is included in the hyperplane at infinity. Let $p_j(\zeta_1, ..., \zeta_n) := P_j(1, \zeta_1, ..., \zeta_n)$ be global sections for the line bundles $[\mathcal{D}_j]$ and ω be a $(q,0)$-rational form in $\mathbb{P}^n(\mathbb{C})$ $(q \le n-m)$, of the form

$$\omega \ := \frac{\sum\limits_{1\le i_1<\cdots<i_q\le n} h_{\mathcal{I}}(\zeta) \bigwedge\limits_{l=1}^{q} d\zeta_{i_l}}{g(\zeta)} = \frac{\sum\limits_{\mathcal{I}} h_{\mathcal{I}}\, d\zeta_{\mathcal{I}}}{g} ,$$

such that

$$\mathrm{codim} \left(\bigcap_{j=1}^{m} \mathrm{Supp}\,[\mathcal{D}_j] \cap \{\text{polar set of}\, \omega\} \right) = m+1 \, .$$

Let $(t_{j0}, ..., t_{jm})_{m+1\le j\le n}$ be local coordinates on $G(m,n)$ about the projective m-plane $\{\zeta_{m+1} = \cdots = \zeta_n = 0\}$ and

$$l_j(t,\zeta) := \zeta_j - t_{j0} - \sum_{k=1}^{m} t_{jk}\zeta_k \, , \ j = m+1, ..., n \, .$$

Here, we assume that the projective m-line defined by the f_j's does not intersect the intersection of $\bigcap\limits_{j=1}^{m} \mathrm{Supp}\,[\mathcal{D}_j]$ with the polar set of the meromorphic form ω.

Let us expand :

$$\bigwedge_{j=m+1}^{n} \partial_{\zeta,t}[l_j(t,\zeta]$$
$$= \sum_{\substack{\mathcal{J}\subset\{m+1,...,n\} \\ \#\mathcal{J}=q}} \ \sum_{(\rho_{m+1},...,\rho_q)\in\{0,...,m\}^q} \epsilon_{\mathcal{J}}\, \zeta_{[\rho]}\, (d_\zeta[l(t,\zeta)])_{\mathcal{J}^c} \wedge dt_{\mathcal{J},\rho} + ...$$

where $\epsilon_{\mathcal{J}} = \pm 1$, $\mathcal{J} = \{j_1, j_2, ..., j_q\}$ is considered in increasing order, as well as $\{m+1, ..., n\} \setminus \mathcal{J} = \{k_1, ..., k_{n-m-q}\}$, $\rho = (\rho_{m+1}, ..., \rho_q) \in \{0, ..., m\}^q$ and

$$\begin{aligned}
(d_\zeta[l(t,\zeta)])_{\mathcal{J}^c} : &= \bigwedge_{\sigma=1}^{n-m-q} d_\zeta[l_{k_\sigma}(t,\zeta)] \\
dt_{\mathcal{J},\rho} : &= \bigwedge_{\tau=1}^{q} dt_{j_\tau, m_\tau} \\
\zeta_{[\rho]} : &= \prod_{\rho_j \in \rho, \rho_j \neq 0} \zeta_{\rho_j} \,.
\end{aligned}$$

The closed expression for the trace of ω relatively to the cycle

$$C(\mathcal{D}) := \sum_\sigma \nu_\sigma(\mathcal{D})\, V_\sigma$$

in terms of multivariate residue calculus is

$$\begin{aligned}
\mathrm{Tr}_{C(\mathcal{D})}[\omega](t) = &\sum_{\substack{\mathcal{J} \subset \{m+1,\ldots,n\} \\ \#\mathcal{J} = q}} \ \sum_{(\rho_{j_1}, \ldots, \rho_{j_q}) \in \{0,\ldots,m\}^q} \epsilon_{\mathcal{J}} \times \\
&\times \mathrm{Res} \begin{bmatrix} \zeta_{[\rho]} \omega \wedge (d_\zeta[l(t,\zeta)])_{\mathcal{J}^c} \wedge \bigwedge_{k=1}^{m} dp_k \\ p_1, ..., p_m, l_{m+1}(t,\zeta), ..., l_n(t,\zeta) \end{bmatrix} dt_{\mathcal{J},\rho} \,. \qquad (1)
\end{aligned}$$

The fact that the $\mathrm{Tr}_{C(\mathcal{D})}[\omega]$ is a rational form whenever ω is rational can be viewed in such a context as a consequence of the classical transformation law in residue calculus.

Let us point out the particular case $m = 1$ and $\mathcal{D} = \mathcal{D}_1$, $\deg \mathcal{D} = D$; let p be a global section of the line bundle $[\mathcal{D}]$ in $\mathbb{C}^n$, that is $p \in \mathbb{C}[\zeta_1, ..., \zeta_n]$, $\deg p = D$ and P be the homogeneization of p ; we assume that the projective line $L_0 := \{\zeta_2 = \cdots = \zeta_n = 0\}$ does not meet the intersection of the support of $\mathcal{D}$ (that is the hypersurface $\{P = 0\}$) with the polar set of ω. Let $(t_{j0}, ..., t_{jm})_{2 \leq j \leq n}$ be local coordinates on $G(1, n)$ about the projective line L_0 and

$$l_j(t,\zeta) := \zeta_j - t_{j0} - t_{j1}\zeta_1 \,, \ j = 2, ..., n \,.$$

If

$$\omega = \frac{\sum_{j=1}^{n} h_j(\zeta) d\zeta_1 \wedge \cdots \wedge \widehat{d\zeta_j} \wedge \cdots \wedge d\zeta_n}{g} \,,$$

let H and G be the respective homogeneizations of the polynomials

$$h := \sum_{k=1}^{n} (-1)^{k-1} h_k \frac{\partial p}{\partial \zeta_k} \quad \text{and} \quad g\,;$$

let also :

$$\tilde{p}(t,\zeta) := \frac{\partial p}{\partial \zeta_1} + \sum_{j=2}^{n} t_{j1} \frac{\partial p}{\partial \zeta_k}$$

and $\widetilde{P}(t,\cdot)$ the homogeneization of $\tilde{p}(t,\cdot)$ respect to the affine variables ζ. Formula (1) becomes in this setting, in a neighborhood of L_0 in $G(1,n)$,

$$\begin{aligned} &\mathrm{Tr}_{C(\mathcal{D})}[\omega] \\ = \ & (-1)^{n-1} \sum_{(\rho_2,\dots,\rho_n)\in\{0,1\}^{n-1}} \mathrm{Res} \begin{bmatrix} \frac{h(\zeta)\zeta_{[\rho]}}{g(\zeta)}\, d\zeta_1 \wedge \cdots \wedge d\zeta_n \\ \\ p(\zeta), l_2(t,\zeta), \dots, l_n(t,\zeta) \end{bmatrix} \bigwedge_{k=2}^{n} dt_{k,\rho_k} \end{aligned} \tag{2}$$

The projective line $L_t := \{l_j(t,\zeta) = 0\,;\ j = 2,\dots,n\}$ intersects the hyperplane at infinity at the point $z_{t,\infty} := [0 : 1 : t_{21} : \cdots : t_{n1}]$; it follows then from the residue formula (on the projective line L_t) that the trace $\mathrm{Tr}_{\mathcal{D}}[\omega]$ equals also :

$$\begin{aligned} &\mathrm{Tr}_{C(\mathcal{D})}[\omega](t) \\ &= (-1)^n \sum_{(\rho_2,\dots,\rho_n)\in\{0,1\}^{n-1}} \mathrm{Res}_{L_t\,;\,z_{t,\infty}} \left[\frac{z_1^{|\rho|} H(z)(z_0\, dP - D\, P\, dz_0)}{P(z)\widetilde{P}(t,z)G(z) z_0^{|\rho|+2+\max\deg h_j - \deg g}} \right] \\ &\qquad\qquad \times \bigwedge_{k=2}^{n} dt_{k,\rho_k}\,, \end{aligned} \tag{3}$$

where $|\rho| = \rho_2 + \cdots + \rho_n$; note in particular that $\mathrm{Tr}_{C(\mathcal{D})}[\omega] \equiv 0$ whenever

$$\max(\deg h_j) \leq \deg g - n - 1\,;$$

this key fact was pointed out by P. Griffiths in [13], section 3.c, when ω is of the particular form

$$\omega = \frac{h_1(\zeta) d\zeta_2 \wedge \cdots \wedge d\zeta_n}{\frac{\partial p}{\partial \zeta_1}(\zeta)}$$

(in such a case $\mathrm{Tr}_{C(\mathcal{D})}[\omega] \equiv 0$ whenever $\deg h_1 \leq \deg p - n - 2$).

As we pointed it out, either the transformation law in multidimensional residue calculus (starting with the closed expressions (1) or (2)), either residue theorem on complex lines (starting with the closed expression (3) in the case $m=1$) show that the trace of any $(q,0)$-rational form in $\mathbb{P}^n(\mathbb{C})$ respect to some $n-m$-purely dimensional cycle defines a rational form on the smooth manifold $G(m,n)$; the fact that trace preserves rationality is a result which is known as Abel's theorem. In fact, it may be used for example in the following context (which is in the spirit of Abel's work), starting with the principal divisor $\mathcal{D}$ in $\mathbb{P}^2(\mathbb{C})$ defined by the homogeneous irreducible polynomial

$$x_2^2 x_0 - 4x_1^3 + g_2(\Lambda) x_1 x_0^2 + g_3(\Lambda) x_0^3 \,,$$

the support of which is an elliptic curve Γ_Λ (associated to some lattice Λ in $\mathbb{C}$) ; such a curve is analytically diffeomorphic to the complex analytic variety $\mathbb{C}/\Lambda$ thanks to the map

$$\mathbb{C}/\Lambda \to \Gamma_\Lambda \;:\; \overline{z} \to [1 : \mathcal{P}(z) : \mathcal{P}'(z)] \,,$$

whose inverse is precisely :

$$p \in \Gamma_\Lambda \to \int_\gamma \omega \,,$$

where ω denotes the rational form dx_1/x_2 and $\gamma \;: [0,1] \to \mathbb{P}^2(\mathbb{C}) \setminus \{x_2 = 0\}$ is a piecewise-smooth path such that $\gamma(0) = [0:0:1]$. The associativity of the group law on Γ_Λ is a consequence of the fact that $\omega \wedge [\Gamma_\Lambda]$ is a $\overline{\partial}$-closed current on $\mathbb{P}^2(\mathbb{C})$ since

$$\frac{d[\mathcal{P}(z)]}{\mathcal{P}'(z)} = dz \,,$$

which is an holomorphic form on $\mathbb{C}$; so the trace of ω (respect to $\mathcal{D}$) is a $\overline{\partial}$-closed rational form on $\mathbb{P}^2(\mathbb{C})^*$, that is identically 0, which implies precisely the associativity of the group law on Γ_Λ. For an introductive presentation to the deep relation between the concept of trace and Abel's work, we invite the reader to consult the exhaustive survey of J. E. Björk [6].

It should be pointed out that it was proved in the analytic context by G. Henkin and M. Passare in [16] that, whenever U is a m-concave open set in $\mathbb{P}^n(\mathbb{C})$, V a closed $n-m$-dimensional irreducible analytic subset in U, then the trace of any meromorphic $(q,0)$ form on V (that is a $(q,0)$ meromorphic form in some neighborhood of V in U, with $q \leq n-m$) is a meromorphic form on U_m^* ; when ω is holomorphic in U, then $\mathrm{Tr}_V[\omega]$ is holomorphic in U_m^*. Note that such a result can be transposed to the context of purely dimensional $n-m$-cycles

$$C(\mathcal{D}) = \sum_\sigma \nu_\sigma(\mathcal{D})\, V_\sigma$$

in U : if ω is a a $(q,0)$-meromorphic form on $\mathrm{Supp}\,[C(\mathcal{D})]$ (that is in a neighborhood of $\mathrm{Supp}\,[C(\mathcal{D})]$ in U) such that

$$\omega \wedge \left(\sum_\sigma \nu_\sigma(\mathcal{D})\ \wedge [V_\sigma]\right)$$

is a $\overline{\partial}$-closed current in U, then $\mathrm{Tr}_{C(\mathcal{D})}[\omega]$ is an holomorphic $(q,0)$-form in U_m^*.

4 About Abel's theorem and its relation with tomography

The work of G. Gindikin and G. Henkin [11, 15] is the main source of inspiration for this section. Abel's theorem is deeply connected with tomography (one should say in fact tomography of "thin" objets, such as closed analytic sets). Here again, the notion of incidence manifold (together with the efficient tool which consists in using Poincaré residue) plays a crucial role.

Let U be some m-concave open subset in $\mathbb{P}^n(\mathbb{C})$ containing the m-line

$$L_0 := \{\zeta_{m+1} = \cdots = \zeta_n = 0\}\,;$$

let φ be a smooth $\overline{\partial}$-closed (n,m)-form in U ; the Abel-Radon transform of φ is a $(n-m,0)$-holomorphic form on U_m^* which is defined by the procedure we will describe below ; we will indicate in fact how such a form is constructed in a neighborhood W^* of L_0 in U_m^*, (a similar construction could be made about any m-line in U_m^*) ; note that the reason why we keep track of homogeneous coordinates in $\mathbb{P}^n(\mathbb{C})$ is that one can expect in the future such construction to be transposed within the frame of complete simplicial toric varieties, playing then with homogeneous coordinates as introduced in [9] or used in [12].

In $U \times W^*$, we will consider the semi-meromorphic differential form :

$$\Omega := \Big([z_0 : \cdots : z_n]\,;\, t_{jk}, j = m+1, ..., n, k = 0, ..., m\Big)$$

$$\to \Big(\bigwedge_{j=m+1}^{n} \partial_{t_j}\Big[\log\Big(\frac{\prod_{j=m+1}^{n} |L_{t,j}(z)|^2}{\|z\|^{2(n-m)}}\Big)\Big]\Big) \wedge \varphi([z_0 : \cdots : z_n])\,,$$

where

$$L_{t,j} := z_j - t_{j,0} z_0 - \sum_{k=1}^{m} t_{j,k} z_k\,,\ j = m+1, ..., n\,;$$

the polar set of such a differential form is the union of the smooth hypersurfaces

$$Y_j := \{([z_0 : \cdots : z_n], t)\,;\, L_{t,j}(z) = 0\}\,, \quad j = m+1, ...n\,,$$

which intersect in a transversal way ; note that $Y_{m+1} \cap \cdots \cap Y_n$ defines precisely the incidence manifold $\mathrm{Icd}_m(U)$ in $U \times W^*$; since Ω is d-closed in $(U \times W^*) \setminus \{Y_{m+1} \cup \cdots \cup Y_n\}$, one can define the iterated Poincaré residue of Ω (following J. Leray's construction), which leads to a (n, m) smooth closed form on the manifold $\mathrm{Icd}_m(U)$. Such a form can be expressed (in the coordinates $[z_0 : ... : z_m], t$ on $\mathrm{Icd}_m(U)$ in $U \times W^*$) as

$$\begin{aligned}&\mathrm{Res}_{Y_n} \circ \cdots \circ \mathrm{Res}_{Y_{m+1}}[\Omega]\\ =& \sum_{(\rho_{m+1},\dots,\rho_n)\in\{0,\dots,m\}^q} \Omega^{(\varphi)}_{(\rho_{m+1},\dots,\rho_n)}([z_0 : \cdots : z_m], t) \wedge \bigwedge_{k=m+1}^{n} dt_{k,\rho_k} + \cdots\end{aligned}$$

where the $\Omega^{(\varphi)}_{(\rho_{m+1},\dots,\rho_n)}$ are (m, m)-smooth forms (depending on the homogeneous coordinates $z_0, ..., z_m$) defining (m, m) forms in U ; one can now define a holomorphic $(n-m, 0)$-form in W^* (which is the *Abel-Radon transform* of φ) as

$$\mathcal{AR}_m[\varphi] := \sum_{(\rho_{m+1},\dots,\rho_n)\in\{0,\dots,m\}^q} \Big(\int_{z\in L_t} \Omega^{(\varphi)}_{(\rho_{m+1},\dots,\rho_n)}(z, t)\Big) \bigwedge_{k=m+1}^{n} dt_{k,\rho_k}\,,$$

where L_t denotes the m-subspace of $\mathbb{P}^n(\mathbb{C})$ defined by the $L_{t,j}$, $j = m+1, ..., n$.

Now is the relation between the Abel and the Abel-Radon tranform : if

$$C(\mathcal{D}) = \sum_{\sigma} \nu_\sigma(\mathcal{D})\, V_\sigma$$

is a $n-m$ analytic cycle in U, and ω is a meromorphic $(n-m,0)$-form on the support of $C(\mathcal{D})$, then the current

$$T_{\omega,\mathcal{D}} := \omega \wedge \Big(\sum_{\sigma} \nu_\sigma(\mathcal{D})\, V_\sigma \Big)$$

defines a (n,m)-$\overline{\partial}$-closed current in the m-concave open set U' which is defined as complement (in U) of the union of m-planes in U_m^* which intersect the $n-m-1$-analytic set

$$\mathrm{Supp}(C(\mathcal{D})) \cap \{\text{polar set of } \omega\Big\}.$$

It follows from Dolbeault's theorem (see for example [14]) that there exists a smooth $\overline{\partial}$-closed (n,m) form $\varphi_{\omega,\mathcal{D}}$ in U' which defines (as a current) the same cohomology class than $T_{\omega,\mathcal{D}}$ in $H^{n,m}(U')$. The Abel-Radon transform of $\varphi_{\omega,\mathcal{D}}$ coincides in $(U')^*_m$ with the Abel transform of ω respect to the analytic cycle $C(\mathcal{D})$ (the last one is defined as a meromorphic form in U^*_m, note that $(U')^*_m$ is the complement of some analytic hypersurface in U^*_m).

It is important to mention here that in the global setting (when $U = \mathbb{P}^n(\mathbb{C})$, $\mathcal{D}$ is defined by homogeneous polynomials $P_1, ..., P_M$, and ω is a rational form in $\mathbb{P}^n(\mathbb{C})$), one could use, in order to express $\varphi_{\omega,\mathcal{D}}$ and therefore $\mathrm{Tr}_{C(\mathcal{D})}[\omega]$ in terms of the P_j's (following the ideas developped in [2] and [3]) the Levine form [17] in $\mathbb{P}^{2n+1}(\mathbb{C})$ (where the homogeneous coordinates are denoted as $[z_0 : \cdots : z_n : w_0 : \cdots : w_n] = [\underline{Z} : \underline{w}]$),

$$\begin{aligned} L(Z,w) \ &:= -\log\Big[\frac{\|\underline{Z}-\underline{w}\|^2}{\|\underline{Z}\|^2 + \|\underline{w}\|^2}\Big] \\ &\quad \times \Big(\sum_{k=0}^{n} (dd^c \log \|\underline{Z}-\underline{w}\|^2)^k \wedge (dd^c \log(\|\underline{Z}\|^2 + \|\underline{w}\|^2))^{n-k} \Big), \end{aligned}$$

where $d^c := (i/2\pi)(\overline{\partial} - \partial))$; if π denotes the map

$$\pi \ : ((\mathbb{C}^{n+1})^*)^2 \times (\mathbb{C}^*)^2 \mapsto (\mathbb{C}^{2(n+1)})^* \ : (\underline{Z}, \underline{w}, (\beta_0, \beta_1)) \mapsto (\beta_0 \underline{Z}, \beta_1 \underline{w})$$

and δ denotes the diagonal in $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$, one has, in this product variety, the equality between currents

$$dd^c \mathcal{Y} + [\delta] = \Theta$$

where Θ is a smooth form and

$$\mathcal{Y}(Z,w) := \int_{\beta\in\mathbb{P}^1(\mathbb{C})} \pi^*(L)(Z,w,\beta)\,;$$

one may introduce a $(n-1,m-1)$ current G defined (formally) as follows :

$$\langle G\,,\,\varphi\rangle := \sum_\sigma \nu_\sigma(\mathcal{D}) \int_{V_\sigma\times\mathbb{P}^n(\mathbb{C})} \mathcal{Y}(\underline{Z},\underline{w})\wedge\omega(\underline{Z})\wedge\varphi(\underline{Z})$$
$$\forall\varphi\in\mathcal{D}^{(1,n-m+1)}(U')$$

(such a current can be constructed explicitely from the homogeneous polynomials $P_1,...,P_M$ defining the cycle $C(\mathcal{D})$, see [3], using for example analytic continuation techniques). One can check that in U',

$$dd^cG + T_{\omega,\mathcal{D}} = \varphi_{\omega,\mathcal{D}}\,,$$

where $\varphi_{\omega,\mathcal{D}}$ is a smooth representant for the cohomology class of $T_{\omega,\mathcal{D}}$ in $H^{n,m}(U')$.

We take the opportunity to notice here how the ideas developped for the construction of the trace (based on the extensive use of the incidence variety with its two projections π_1 and π_2) and the construction of normalized Green currents attached to algebraic cycles in $\mathbb{P}^n(\mathbb{C})$ (based on the extensive use of the duplicated space $\mathbb{P}^n(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})$, together with the Levine form for the diagonal) are intrinsically similar. The major difference is that trace constructions are deeply related with residue theory (as shown in the preceeding sections), that is division theory, while construction of normalized Green currents deals with intersection theory. This distinction between intersection theory (linked with the theory of integration currents) and division theory (where factorization of integration currents, that is residue theory, plays a fundamental role) remains indeed the permanent guideline of this survey talk.

5 About Abel's inverse theorem

An interesting fact that can be viewed as a consequence of the G.A.G.A principle [19] in the algebraic case is that Abel's theorem admits a converse which can be stated as follows in the analytic setting :

Théorème 5.1 (Henkin-Passare) [16] *Let U be some m-concave open set in $\mathbb{P}^n(\mathbb{C})$ such that $L_0 = \{\zeta_{m+1} = \cdots = \zeta_n = 0\} \subset U$, $a_1, ..., a_d$ d distinct points on the m-plane L_0 in $\mathbb{P}^n(\mathbb{C})$, and $V_1, ..., V_d$ d germs of smooth $n-m$-manifolds respectively in $(\mathbb{P}^n(\mathbb{C}), a_1)$,...,$(\mathbb{P}^n(\mathbb{C}), a_d)$, defined as smooth complete intersections*

$$V_j := \{f_{j1} = \cdots = f_{jm} = 0\}\,,\ f_{jk} \in \mathcal{O}_{a_j}$$
$$\left[\bigwedge_{k=1}^{m} df_{jk} \wedge d\zeta_{m+1} \wedge \cdots \wedge d\zeta_n\right](a_j) \neq 0$$

for $j = 1, ..., d$. Assume that for some choice of germs of $(n-m, 0)$-forms $\psi_1, ..., \psi_d$ respectively at $a_1, ..., a_d$, the germ

$$t \stackrel{\Psi}{\to}$$
$$\sum_{(\rho_{m+1},\dots,\rho_n)\in\{0,\dots,m\}^q} \sum_{j=1}^{d} \mathrm{Res}\left[\begin{array}{c} \zeta_{[\rho]}\,\psi_j \wedge \bigwedge_{k=1}^{m} df_{jk} \\ f_{j1}, ..., f_{jm}, l_{m+1}(t,\zeta), ..., l_n(t,\zeta) \end{array}\right] \bigwedge_{m+1}^{n} dt_{k,\rho_k}$$

is a germ of meromorphic form in U_m^* ; then one can find

- *a closed $n-m$ analytic set V in U such that $V \cap L_0 = \{a_1, ..., a_d\}$ and $V_j \subset V$ for $j = 1, ..., d$;*
- *a meromorphic form ψ on V in U such that*

  $$\mathrm{Tr}[\omega](t) = \Psi(t)$$

  in a neighborhood of L_0 in U_m^*.

The proof of this theorem can be reduced in fact to the case $m = 1$. In the case $U = \mathbb{P}^n(\mathbb{C})$, $m = 1$, one should mention in the same spirit a result by J. Wood [21] :

Théorème 5.2 (J. Wood) *Let L_0, $a_1, ..., a_d$, $V_1, ..., V_d$, $f_j = f_{j1}$, $j = 1, ..., d$, as above ; a necessary and sufficient condition in order that there exists a polynomial p (with degree d) (that can be explicitely constructed) such that the algebraic hypersurface $\{p = 0\}$ interpolates the germs V_j, $j = 1, ..., d$,* *and is such that L_0 does intersects the corresponding projective hypersurface $\{P=0\}$ only at the points $a_1,...,a_d$, is that the function*

$$t \to \sum_{j=1}^{d} \text{Res} \begin{bmatrix} \zeta_1 \bigwedge_{j=2}^{m} d_\zeta[l_j(t,\zeta)] \wedge df_j \\ f_j, l_2(t,\zeta), ..., l_n(t,\zeta) \end{bmatrix}$$

is polynomial (with degree at most one) in the variables $t_{20},...,t_{2n}$.

Wood's theorem involves some hypothesis about the trace of a $(0,0)$-form (namely the forme ζ_1) respect to some union of germs of hypersurfaces, while Abel's inverse theorem (in the case $m=1$) involves some hypothesis about the trace of some $(n-1,0)$-form (namely the form defined near each a_j as ψ_j) respect to some union of germs of hypersurfaces.

The advantage of Wood's result is that it provides an explicit construction of the polynomial p in terms of the trace of the form ζ_1 ; this is not the case in the proofs of Abel's inversion theorem (see [13], when the trace is zero, [16] when the trace is rational) .

A second motivation to relate Abel's inverse theorem and Wood's result is that it would give some insight about the following natural question : starting with the hypothesis as in Henkin-Passare's theorem, what can be said if for some choice of germs of $(n-m,0)$-forms $\psi_1,...,\psi_d$ respectively at $a_1,...,a_d$, the germ

$$t \overset{\Psi}{\to}$$

$$\sum_{(\rho_{m+1},...,\rho_n)\in\{0,...,m\}^q} \sum_{j=1}^{d} \text{Res} \begin{bmatrix} \zeta^\rho \psi_j \wedge \bigwedge_{k=1}^{m} df_{jk} \\ f_{j1},...,f_{jm}, l_{m+1}(t,\zeta),...,l_n(t,\zeta) \end{bmatrix} \bigwedge_{k=m+1}^{n} dt_{k,\rho_k}$$

is a germ in U_m^* which coefficients at any point $z_0 \in U_m^*$ are algebraic over $\mathcal{O}_{z_0}$ (or are algebraic forms when $U=\mathbb{P}^n(\mathbb{C})$) ? Such a question has been studied by S. Collion in [8].

Even in the case when $U=\mathbb{P}^n(\mathbb{C})$, $d=1$, $a_1=\underline{0}$ and V_1 is defined in a neighborhood of the origin as

$$V_1 := \{\zeta_1 = \varphi_1(\zeta_2,...,\zeta_n) \,;\, \max_{2\le j\le n} |\zeta_j| \le \epsilon\} \,,$$

where φ_1 is holomorphic in the polydisc $\Delta_{n-1}(\underline{0}\,;\epsilon,...,\epsilon)$ for ϵ sufficiently small with $\varphi_1(\underline{0})=0$, proving Abel's inverse theorem (or verifying the validity of Wood's criterion, which is the same) in such a particular situation is not immediate : it amounts to prove that φ_1 is necessarily a linear function when the trace of some $n-1$ non identically zero differential form

$$\psi_1 = \psi(\zeta_2, ..., \zeta_n)\, d\zeta_2 \wedge \cdots \wedge d\zeta_n$$

(with ψ is holomorphic $\Delta_{n-1}(\underline{0}\,;\,\epsilon, ..., \epsilon)$), namely

$$t \to \sum_{(\rho_2,\ldots,\rho_n)\in\{0,1\}^q} \mathrm{Res}\begin{bmatrix} \zeta^\rho \psi(\zeta_2, ..., \zeta_n)\, d\zeta_1 \wedge \cdots \wedge d\zeta_n \\ \\ \zeta_1 - \varphi_1(\zeta_2, ..., \zeta_n), l_2(t,\zeta), ..., l_n(t,\zeta) \end{bmatrix} \bigwedge_2^n dt_{k,\rho_k}$$

can be continued as a meromorphic form in $G(1,m)$. The simple ideas developped in order to prove directly such a result (such as done recently by Martin Weimann) may be also transposed to the situation $a_1 = \cdots = a_d = \underline{0}$, V_j being described as

$$V_j := \{\zeta_1 = \varphi_j(\zeta_2, ..., \zeta_n)\,;\, \max_{2\le j\le n} |\zeta_j| \le \epsilon\}\,,\ j = 1, ..., d\,,$$

where $\varphi_j(\underline{0}) = 0$, $j = 1, ..., d$ and $\varphi_k \not\equiv \varphi_l$ for $k \neq l$; one can use in this case Lagrange interpolation formula (see for example section 5 in [20]) in order to deduce from the existence of a meromorphic form $\psi(\zeta_2, ..., \zeta_n) d\zeta_2 \wedge \cdots \wedge d\zeta_n$ whose trace is rational the fact that the ψ_j's are algebraic functions. Of course, such a situation does not fit with the standard hypothesis for Abel's inverse theorem (since the a_j's are all equal) but it deserves to be dealt with directly.

In the same vein, it seems interesting to caracterize $n-1$ differential forms on $G(1,n)$ (in a neighborhood of L_0) which are obtained through the trace construction from a collection of germs of hypersurfaces $V_1, ..., V_s$ (non necesseraly smooth) which intersect L_0 at respective distinct points $a_1, ..., a_s$ and are such that the projection $\theta\ : (\zeta_1, \zeta_2, ..., \zeta_n) \in V_1 \cup \cdots \cup V_s \to (\zeta_2, \cdots, \zeta_n)$ is proper. In such a situation, one can prove, using elementary residue theory and Lagrange interpolation, the following : a $n-1$-meromorphic germ of form $\widetilde{\omega}$ (expressed in coordinates (t_{0j}, t_{1j}), $j = 2, ..., n$) at $L_0 \sim \{\underline{t} = \underline{0}\}$ in $G(1,n)$ is the trace of a germ of meromorphic form respect to $V = V_1 \cup \cdots \cup V_s$ if and only if there are two polynomials in $\mathcal{M}(t)[X]$,

$$\begin{aligned} F(t,X) &= t^d - \sigma_1(t) X^{d-1} + \cdots + (-1)^d \sigma_d(t) \\ H(t,X) &= \gamma_0(t) X^{d-1} + \cdots + \gamma_{d-1}(t) \end{aligned}$$

which satisfy the Burgers type equations

$$\begin{aligned} X\frac{\partial F}{\partial t_{j0}} &\equiv \frac{\partial F}{\partial t_{j1}} \pmod{F} \\ X\frac{\partial H}{\partial t_{j0}} &\equiv \frac{\partial H}{\partial t_{j1}} \pmod{F} \end{aligned}$$

and are such that

$$\tilde{\omega}(t) = \mathrm{Res}_X \begin{bmatrix} dX \wedge H(t,X)\Psi(t,X) \\ \\ F(t,X) \end{bmatrix}, \tag{4}$$

where

$$\Psi(t,X) := \left(F'_X(t,X) - \sum_{j=2}^{n} t_{j1}\frac{\partial F}{\partial t_{j0}}(t,X)\right) \bigwedge_{j=2}^{n} \left(dt_{j0} + X\, dt_{j1}\right)$$

(the residue symbol in (4) being a residue symbol in one variable). It remains a challenge (which has been recently affronted by Martin Weimann, a PHD student in Bordeaux) to proof that whenever (for fixed d) there exists such a germ of $n-1$ form $\tilde{\omega}$ with algebraic coefficients, then all functions σ_j and γ_j involved in a choice of F and H which fits with (4) may be chosen as germs of algebraic functions (or, in case $\tilde{\omega}$ is rational, that Wood's criterion with d germs of hypersurfaces is fulfilled).

Another interesting related question seems the following problem : assume that $a_1, ..., a_d$ are distinct points on the line $L_0 := \{\zeta_1 = 0\}$ in $\mathbb{C}^n$ and that $V_1, ..., V_d$ are d germs of hypersurface respectively at the points $a_1, ..., a_d$ such that the tangent spaces $T_{a_j}(V_j)$, $j = 1, ..., d$, coincide (along some hyperplane $\xi \in (\mathbb{P}^n(\mathbb{C}))^*$) ; again the situation here does not fit with the hypothesis of Abel's inverse theorem since L_0 does not intersect transversally the germs of hypersurfaces V_j. Nevertheless, one can introduce the dual germs $\check{V}_j$, $j = 1, ..., d$, which now are smooth germs of hypersurfaces intersecting at ξ in $(\mathbb{P}^n(\mathbb{C}))^*$; can one use projective duality (see the next section) in order to derive directly Abel's inverse theorem in such a setting (or some other formulation for it, which would be precisely the dual formulation of Abel's inverse theorem formulated in $(\mathbb{P}^n(\mathbb{C}))^*$) ?

References

[1] D. Barlet, Le faisceau $\omega_X^\bullet$ sur un espace analytique X de dimension pure, Lecture Notes in Math. 670, Springer-Verlag, 1978, 187–204.

[2] C. A. Berenstein, A. Yger, Green currents and analytic continuation, Journal d'Analyse Mathématique 75, 1998, 1-50.

[3] C. A. Berenstein, A. Yger, Residue currents, integration currents in the non complete intersection case, J. reine. angew. Math. 527, 2000, 203-235.

[4] C. A. Berenstein, A. Vidras, A. Yger, Analytic residues along algebraic cycles, *submitted* to the proceedings of the conference FOCM'02, University of Minnesota, August 2002.

[5] J. E. Björk, Residue currents and $\mathcal{D}$-modules on complex manifolds, *preprint*, Stockholm, 1996.

[6] Residues and $\mathcal{D}$-modules, a tribute to the the Abel bicentennial conference, 2002, *to appear* in the conference acts.

[7] N. Coleff, M. Herrera, *Les courants résiduels associés à une forme méromorphe*, Springer-Verlag (Lecture Notes in Math. 633), Berlin, New-York, 1978.

[8] S. Collion, Transformation d'Abel et formes différentielles algébriques, Comptes Rendus Acad. Sci. Paris, 323, Série I, 1237-1242, 1996.

[9] D. Cox, The homogeneous coordinate ring of a toric varirty, J. Algebraic geometry 4, 1995, 17-50.

[10] B. Fabre, Nouvelles variations sur les théorèmes d'Abel et Lie, Doctorat d'Université soutenu à Paris VI le 4 décembre 2000.

[11] G. Gindikin, G. Henkin, Integral geometry for $\overline{\partial}$ cohomology in q-linear concave domains in $\mathbb{CP}^n$, Funkst. Anal. i Pril. 12 (1978), 6-23.

[12] I. M. Gelfand, M. M. Krapanov and A. V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[13] P. A. Griffiths, Variations on a Theorem of Abel, Inventiones math. 35 (1976), 321-390.

[14] P. Griffiths, J. Harris, Principles of algebraic geometry, Wiley Interscience, New York, 1978.

[15] G. Henkin, The Abel-Radon transform and several complex variables, *Proceedings of the conference in honor of Guning and Kohn*, Princeton, 1992, Ann. of Math. Studies, 137 (1995), 223-275.

[16] M. Passare, G. Henkin, Abelian differentials on singular varieties and variations on a theorem of Lie-Griffiths, Inventiones math. 135, 1999, 297-328.

[17] H. Levine, A theorem on holomorphic mappings into complex projective space, Ann. of Math. 71 (1960), 529-535.

[18] M. Méo, Résidus dans le cas non nécessairement intersection complète, C. R. Acad. Sci. Paris Sér. I Math. 333 (1), 2001, 33–38.

[19] J. P. Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, 6, 1955-56, 1-42.

[20] A. Tsikh, A. Yger, Residue currents, *to appear in* Encyclopedia in Mathematics, Nauka, Moscow, 2003.

[21] J. Wood, A simple criterion for local hypersurfaces to be algebraic, Duke Mathematical Journal, 51, 1, 1984, 235-237.