On asymptotic approximations of the residual currents *

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Abstract

We use a $\mathcal{D}$-module approach to discuss positive examples for the existence of the unrestricted limit of the integrals involved in the approximation to the Coleff-Herrera residual currents (in the complete intersection case.) Our results provide also asymptotic developments for these integrals.

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1 Introduction.

Let $f_1, \dots, f_p$ be p holomorphic functions in a neighborhood V of the origin in $\mathbf{C}^n$ ($p \leq n$), defining in this neighborhood a complete intersection. It is known from [12] that the limits

$$\lim_{\delta \mapsto 0} \frac{1}{(2\pi i)^p} \int\limits_{\substack{|f_j(\zeta)|=\epsilon_j(\delta) \\ 1 \leq j \leq p}} \frac{\varphi}{f_1 \dots f_p}, \ \varphi \in \mathcal{D}^{n,n-p}(V) \tag{1.1}$$

exist when $\delta \mapsto (\epsilon_1(\delta), \dots, \epsilon_p(\delta))$ is an admissible path, that is

$$\lim_{\delta \mapsto 0} \frac{\epsilon_j(\delta)}{\epsilon_{j+1}^m(\delta)} = 0 \text{ for any } j \in \{1, \dots, p-1\} \text{ and any } m \in \mathbf{N}. \tag{1.2}$$

The semianalytic chain $\{|f_1| = \epsilon_1, \dots, |f_1| = \epsilon_p\}$ is oriented here as the Shilov boundary $\{|\zeta_1| = \epsilon_1, \dots, |\zeta_p| = \epsilon_p\}$ of the polydisk as in the usual Cauchy formula (see [13] , chapter 6.) Moreover, it was shown in [12] that the above limit (1.1) does not depend on the admissible path but just (in an alternating way) on the ordering of the indexation for $f_1, \dots, f_p$. Moreover it defines a $(0, p)$ current on V denoted as

$$\varphi \mapsto < \bar{\partial}\frac{1}{f_1} \wedge \dots \wedge \bar{\partial}\frac{1}{f_p}, \varphi > \tag{1.3}$$

When φ is a $\bar{\partial}$-closed $(n, n-p)$ form, it follows from the Stokes' formula that the almost everywhere defined function

$$(\epsilon_1, \dots, \epsilon_p) \mapsto I(\epsilon_1, \dots, \epsilon_p; \varphi) = \frac{1}{(2\pi i)^p} \int\limits_{\substack{|f_j(\zeta)|=\epsilon_j \\ 1 \leq j \leq p}} \frac{\varphi}{f_1 \dots f_p} \tag{1.4}$$

is constant for $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_p)$ close to $\underline{0}$, and therefore admits trivially a limit when $\underline{\epsilon}$ tends to $\underline{0}$. The question that arises naturally is whether the unrestricted limit

$$\lim_{\underline{\epsilon} \mapsto \underline{0}} \int\limits_{\substack{|f_j(\zeta)|=\epsilon_j \\ 1 \leq j \leq p}} \frac{\varphi}{f_1 \dots f_p} \tag{1.5}$$

(the right hand side being almost everywhere defined by Sard's theorem) exists when φ is an arbitrary element in $\mathcal{D}^{n,n-p}(V)$.

A counterexample due to M.Passare and A.Tsikh in [18] gives a negative answer to this question, even when $p = 2$, and f_1, f_2 define the origin as an isolated zero. It fails for example for the mapping defined by

$$\begin{array}{ccc} (\mathbf{C}^2, 0) & \stackrel{(f_1,f_2)}{\longrightarrow} & (\mathbf{C}^2, 0) \\ (z_1, z_2) & \longrightarrow & (z_1^4, z_1^2 + z_2^2 + z_1^3)\,. \end{array} \tag{1.6}$$

More striking counterexamples have been given recently by J. E. Björk in [10], section 7.2. The unrestricted continuity of (1.5) at the origin is not true for the map

$$\begin{array}{ccc} (\mathbf{C}^2, 0) & \stackrel{(f_1,f_2)}{\longrightarrow} & (\mathbf{C}^2, 0) \\ (z_1, z_2) & \longrightarrow & (z_1^m, z_2^3 + z_1 + z_1^2)\,, \end{array} \tag{1.7}$$

where m is any strictly positive integer. In the last example note that one has $df_2(0) \neq 0$, so that the answer to the question when $n = p = 2$ may be negative even if one of the functions (f_1, f_2) is a coordinate!
The existence of such a rich family of counterexamples motivates the search for positive cases. In this direction J. E. Björk proved in [10], section 7.3, that for $p = n = 2$, the unrestricted limit (1.5) exists when f_1, f_2 are homogeneous polynomials.
When $p = 1$, there is no problem for the existence of the unrestricted limit [12]. Furthermore, in this case, we have a much more precise result. One can show that for any $\varphi \in \mathcal{D}^{(n,n-1)}(V)$, the function

$$\epsilon \longrightarrow \frac{1}{2\pi i} \int\limits_{|f|=\epsilon} \frac{\varphi}{f}$$

admits an asymptotic development in the basis $(1, \epsilon^\alpha (\log \epsilon)^\beta)$, $\alpha \in \mathbf{Q}^{+*}$, $\beta \in \mathbf{N}$. This is a consequence of the fact that the sheaf $\mathcal{D}_V[\lambda] f^\lambda$ is coherent as a $\mathcal{D}_V$-module (see [9], theorem 6.1.9.) Such a coherence property implies (see [14]) the existence of an operator of the form

$$\lambda^M - \sum_{k=1}^{M} \lambda^{M-k} \mathcal{Q}_k(z, \partial) \tag{1.8}$$

that annihilates f^λ. As a consequence, we get the rapid decrease of the analytic continuation of the function

$$\lambda \mapsto J(\lambda;\varphi) := \lambda \int |f|^{2(\lambda-1)}\overline{\partial f} \wedge \varphi = \lambda \int_0^\infty s^{\lambda-1} I(s;\varphi) ds$$

on vertical lines $\gamma + i\mathbf{R}$. This result, combined with the fact that the roots of the Bernstein-Sato polynomial are strictly negative rational numbers [14] and with the classical formula for the inversion of the Mellin-Transform, shows (as it was pointed out by J. E. Björk) the existence of an asymptotic development in the sense of the Barlet-Maire [1, 2] for the function

$$\epsilon \longrightarrow I(\varphi;\epsilon) = \frac{1}{2\pi i} \int_{|f|=\epsilon} \frac{\varphi}{f} \text{ when } \varphi \in \mathcal{D}^{n,n-1}(V)\,.$$

One just needs to move to the left, step by step (thanks to the Cauchy formula) the line integral

$$\frac{1}{2i\pi} \int_{\gamma+i\mathbf{R}} \frac{J(\lambda;\varphi)}{\lambda} \epsilon^{-\lambda} d\lambda.$$

In this paper we will give sufficient conditions which ensure the rapid decrease on the vertical lines $\underline{\gamma} + i\mathbf{R}^p$ ($\underline{\gamma} := (\gamma_1,\ldots,\gamma_p) \in \mathbf{R}^p$) of the analytic continuation of the function

$$\begin{aligned} \underline{\lambda} \in \mathbf{C}^p \overset{J(\cdot;\varphi)}{\longrightarrow} & \tfrac{(-1)^{p(p-1)/2}\lambda_1\ldots\lambda_p}{(2i\pi)^p} \textstyle\int |f_1|^{2(\lambda_1-1)} \ldots |f_p|^{2(\lambda_p-1)} \overline{\partial f_1} \wedge \ldots \wedge \overline{\partial f_p} \\ &= \lambda_1 \ldots \lambda_p \int_{[0,\infty[^p} s_1^{\lambda_1-1} \ldots s_p^{\lambda_p-1} I(s;\varphi) ds. \end{aligned} \tag{1.9}$$

The natural sufficient condition for that is the coherence of the $\mathcal{D}_V$-sheaf of modules $\mathcal{D}_V[\lambda_1,\ldots,\lambda_p] f_1^{\lambda_1} \cdots f_p^{\lambda_p}$. Such a condition is for example fullfilled when $(f_1,\ldots,f_p)$ define a morphism without blowing up in codimension 0, with the additional hypothesis

$$df_1 \wedge \ldots \wedge df_p \ = 0 \text{ implies } f_1 \cdots f_p = 0\,. \tag{1.10}$$

This happens for example when $(f_1,\ldots,f_p)$ define an isolated singularity at the origin together with the additional hypothesis (1.10). Such a condition is also fullfilled for examples of the following form

$$\begin{array}{rcl} f:\ (\mathbf{C}^3,0) & \longrightarrow & (\mathbf{C}^2,0) \\ (z_1,z_2,z_3) & \longrightarrow & (z_1^2-z_2^2z_3, z_2) \end{array} \tag{1.11}$$

introduced in [7], section 3.1 (here there is a nonisolated singularity.)
When the coherence condition is valid, the unrestricted limit (1.5) exists. This shows that Björk's example (1.7) appears as an example where the module $\mathcal{D}_{\mathbf{C}^n,0}[\lambda_1,\lambda_2]z_1^{\lambda_1}f_2^{\lambda_2}$ fails to be of finite type as a $\mathcal{D}_{\mathbf{C}^n,0}$-module.
We will also deduce that under such hypothesis, there is an asymptotic development with respect to the basis of functions $(1,\tau^\alpha(\log\tau)^\beta),\ \alpha\in\mathbf{Q}^{+*},\beta\in\mathbf{N}$ for the function

$$\tau \longrightarrow \Theta(\tau;\varphi) := \frac{(-1)^{\frac{p(p-1)}{2}}}{(2\pi i)^p}p!\tau\int_V \frac{\overline{\partial f_1}\wedge\ldots\wedge\overline{\partial f_p}\wedge\varphi}{(\sum\limits_{j=1}^p|f_j|^2+\tau)^{p+1}} \tag{1.12}$$

which satisfies also [20] the equality

$$\Theta(\underline{0};\varphi) =< \bar\partial\frac{1}{f},\varphi> .$$

Moreover, when $p=2$, using the results of C.Sabbah [21], we will interpret this result in terms of geometric invariants related to the discriminant of (f_1,f_2) as a germ of curve in $(\mathbf{C}^2,0)$.
The organization of the paper will be as follows; in section 2, we will recall a few basic results related to b-functions associated to a system of germs $(f_1,\ldots,f_p)$ in ${}_n\mathcal{O}$ defining a germ of complete intersection. Such b-functions will provide us with some way to express the analytic continuation of the function

$$\underline{\lambda}=(\lambda_1,\ldots,\lambda_p)\longrightarrow J(\underline{\lambda};\varphi)$$

from the half plane $\{\Re\lambda_1>1,\ldots,\Re\lambda_p>1\}$ to a meromorphic function in $\mathbf{C}^p$. In §3 we will analyze under which condition one can find a system of Kashiwara operators of the form (1.8) which annihilate $f_1^{\lambda_1}\ldots f_p^{\lambda_p}$. Finally, in §4 we will prove some positive results with respect to the existence of the unrestricted limit (1.5). In the final section we will study the possibility to get an asymptotic development for the function $\tau\mapsto\Theta(\tau;\varphi)$ in (1.12).

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2 About b-functions.

The existence of functional equations of the Bernstein-Sato type for the products $f_1^{\lambda_1} \dots f_p^{\lambda_p}$ was proved simultaneously by C. Sabbah in [21] and by B. Lichtin in [16]. Given a collection of germs of holomorphic functions $f_1, \dots, f_p$ in ${}_n\mathcal{O}$, there is a finite set $\mathcal{L}$ of linear forms with coefficients in $\mathbf{N}$ jointly coprime, and a collection $(b_L)_{L\in\mathcal{L}}$ of polynomials in one complex variable, together with p operators $\mathcal{Q}_1, \dots, \mathcal{Q}_p$ in $\mathcal{D}_{\mathbf{C}^n,0}[\lambda_1, \dots, \lambda_p]$ such that

$$\prod_{L\in\mathcal{L}} b_L(L(\underline{\lambda}))f_1^{\lambda_1} \cdots f_p^{\lambda_p} = \mathcal{Q}_k(\underline{\lambda})[f_1^{\lambda_1} \cdots f_k^{\lambda_k+1} \cdots f_p^{\lambda_p}],\ k = 1, \dots, p\,. \quad (2.1)$$

As soon as we have a set of relations of the form (2.1), we deduce by a standard iteration an identity of the following type

$$B(\lambda_1, \dots, \lambda_p)f_1^{\lambda_1} \cdots f_p^{\lambda_p} = \mathcal{Q}(\lambda)[f_1^{\lambda_1+1} \cdots f_p^{\lambda_p+1}], \quad (2.2)$$

where $\mathcal{Q} \in \mathcal{D}_{\mathbf{C}^n,0}[\lambda_1, \dots, \lambda_p]$ and $B(\underline{\lambda})$ is

$$B(\underline{\lambda}) = b(\lambda_1, \dots, \lambda_p)b(\lambda_1 + 1, \dots, \lambda_p) \dots b(\lambda_1 + 1, \dots, \lambda_p + 1) \quad (2.3)$$

where $b(\underline{\lambda}) = \prod_{L\in\mathcal{L}} b_L(L(\underline{\lambda}))$ as in (2.1). A relation of the form (2.2) is usually known as a Bernstein- Sato relation for $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$. When $p = 1$, we know that the ideal of the polynomials $B(\lambda)$ involved in any relation of the form (2.2) is principal and admits a generator called a Bernstein-Sato polynomial, which has $\lambda + 1$ as a factor. When $p > 1$, both properties fail in general, the ideal of such B is not principal, and there is no reason why $\lambda_i + 1$, for $i = 1, \dots, p$, should divide such a polynomial B. Consider for example the case $n = p = 2$, and take $f_1(z_1, z_2) = z_1^{\alpha_1} z_2^{\beta_1}$, $f_2(z_1, z_2) = z_1^{\alpha_2} z_2^{\beta_2}$, $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$. Nevertheless, it is of some interest to point out that for $p = 2$ we have the following

Proposition 2.1 *Let $f_1, f_2 \in {}_n\mathcal{O}$ define a germ of complete intersection. Then any polynomial $B(\lambda_1, \lambda_2)$ involved in a Bernstein-Sato relation for $f_1^{\lambda_1} f_2^{\lambda_2}$ admits $\lambda_1 + 1, \lambda_2 + 1$ as factors.*

Proof. The proof curiously follows from the nontriviality of the Coleff-Herrera residual current. Let us take some representatives for f_1, f_2 defined in a neighborhood V of the origin where we have a Bernstein-Sato relation of the form (2.2) for some B. Then, by the local duality theorem [22], there is some element $\varphi \in \mathcal{D}^{n,n-2}(V)$ such that

$$< \bar{\partial}\frac{1}{f_1} \wedge \bar{\partial}\frac{1}{f_2}, \varphi >\neq 0.$$

We also know from [4], section 5, that for such a form φ, the function $\widetilde{J}$ defined by

$$\underline{\lambda} \xrightarrow{\widetilde{J}} J(\lambda_1 + 1, \lambda_2 + 1; \varphi) = \frac{(\lambda_1 + 1)(\lambda_2 + 1)}{4\pi^2} \int |f_1|^{2\lambda_1} |f_2|^{2\lambda_2} \overline{\partial f_1} \wedge \overline{\partial f_2} \wedge \varphi$$

is holomorphic in a product of half-planes

$$\{\Re\lambda_1 > -1 - \epsilon, \, \Re\lambda_2 > -1 - \epsilon\}$$

for $\epsilon > 0$ sufficiently small. From the functional equation (2.2) used twice ($\bar{B}$ denotes the polynomial obtained from B after conjugation of all coefficients), it follows that

$$\widetilde{J}(\lambda_1, \lambda_2) = \frac{(\lambda_1 + 1)(\lambda_2 + 1)}{4\pi^2 B(\underline{\lambda})\bar{B}(\underline{\lambda})} \int |f_1|^{2(\lambda_1+1)} |f_2|^{2(\lambda_2+1)} \psi \tag{2.4}$$

for some $\psi \in \mathcal{D}^{(n,n)}(V)$. We now consider the identity (2.4) near the critical point $(-1, -1)$. From the Gauss lemma in the factorial ring ${}_n\mathcal{O}_{(-1,-1)}$, any irreducible factor of $B(\underline{\lambda})$ or of $\bar{B}(\underline{\lambda})$ distinct from $(\lambda_1 + 1)$ or $(\lambda_2 + 1)$ has to divide the holomorphic function

$$(\lambda_1, \lambda_2) \longrightarrow \int |f_1|^{2(\lambda_1+1)} |f_2|^{2(\lambda_2+1)} \psi.$$

Suppose now that $(\lambda_1 + 1)$ does not divide $B(\underline{\lambda})$. Then it does not divide $\bar{B}(\underline{\lambda})$ either. Therefore $B(\underline{\lambda})\bar{B}(\underline{\lambda})$ necessarily divides

$$(\lambda_2 + 1) \int |f_1|^{2(\lambda_1+1)} |f_2|^{2(\lambda_2+1)} \psi \, ,$$

so that in this case, we have near $(-1,-1)$, the following identity

$$\widetilde{J}(\lambda_1,\lambda_2)=(\lambda_1+1)\widehat{J}(\lambda_1,\lambda_2),$$

where $\widehat{J}$ is a holomorphic function. Therefore, we would have

$$J(\underline{0};\varphi)=<\bar{\partial}\frac{1}{f_1}\wedge\bar{\partial}\frac{1}{f_2},\varphi>=0\,,$$

which is a contradiction. So (λ_1+1) divides $B(\underline{\lambda})$ and so does (λ_2+1). The proof is complete. $\diamondsuit$

Dealing with the meromorphic continuation of currents instead of distributions, there may be cancellation of some polar divisors. Such is the case for the function $\underline{\lambda}\mapsto J(\underline{\lambda};\varphi)$ we are interested in. We recall from [4], Proposition 3.6 and Proposition 3.18, the following

Proposition 2.2 *Let f_1, f_2 be two holomorphic functions in n-variables in a neighborhood V of the origin. Then, for any $\varphi\in\mathcal{D}^{n,n-2}(V)$, the polar set of the function*

$$\underline{\lambda}\longrightarrow J(\underline{\lambda};\varphi)$$

in included in a union of hyperplanes (independent of φ) of the form

$$m_{L,1}(\lambda_1+k)+m_{L,2}(\lambda_2+k)+m_{L,0}=0\,,k\in\mathbf{N}^*\,, \tag{2.5}$$

where the vectors $(m_{L,0},m_{L,1},m_{L,2})$, $L\in\mathcal{L}$, lie in a finite subset of $\mathbf{N}^3$ (indexed by $\mathcal{L}$) with $m_{L,1}$, $m_{L,2}\in\mathbf{N}$, and $m_{L,0}\in\mathbf{N}^*$ for any L.

Remark 2.1. The proposition implies that if we write

$$J(\underline{\lambda};\varphi)=\frac{\lambda_1\lambda_2}{4\pi^2B(\underline{\lambda}-\underline{1})\bar{B}(\underline{\lambda}-\underline{1})}\int|f_1|^{2\lambda_1}|f_2|^{2\lambda_2}\psi,$$

then all the factors of $B(\underline{\lambda}-\underline{1})\bar{B}(\underline{\lambda}-\underline{1})$ which are different from λ_1,λ_2 and not of the form (2.5) necesserilly divide in $\{\Re\lambda_1>-\epsilon,\ \Re\lambda_2>-\epsilon\}$ the holomorphic function

$$\underline{\lambda}\longrightarrow\int_V\|f_1\|^{2\lambda_1}\|f_2\|^{2\lambda_2}\psi.$$

We conclude this section with a direct analogue of Kashiwara's theorem about the rationality of the roots of the Bernstein-Sato polynomial in the case where f^λ is replaced by $f_1^{\lambda_1} \dots f_p^{\lambda_p}$ and $f_1, \dots, f_p$ define what is known as a minimal defining system. Let us state the definition, originally introduced by A.Tsikh in [22].

Definition 2.1 Let $f_1, \dots, f_p$ be p holomorphic functions in an open neighborhood $V \subset \mathbf{C}^n$ of the origin so that $f_j(0) = 0$ for every $j = 1, \dots, p$. Assume also that the collection $\{f_1, \dots f_p\}$ defines a complete intersection, that is, the analytic set

$$A = f^{-1}(0) = \bigcap_{j=1}^{p} \{z \in \mathbf{C}^n\,,\ f_j(z) = 0\}$$

has dimension $n - p$. The system $\{f_1, \dots, f_p\}$ is called a *minimal defining system* if and only if the set

$$\mathcal{S}ing\,(A) := \{z \in A\,,\ df(z) := df_1 \wedge \dots \wedge df_p(z) = 0\}$$

is a nowhere dense subset of A.

Remark 2.2. If $f = (f_1, \dots, f_p)$ is a minimal defining system, the set $\mathcal{S}ing\,(A)$ coincides *exactly* with the set of singular points of the analytic set $A = f^{-1}(0)$ (which justifies our terminology); in particular, the set of singular points of the analytic set $f^{-1}(0)$ is in this case a closed analytic subvariety (which is not true in general for an arbitrary analytic set.)

Example 2.1. Let $(f_1, \dots, f_p) : V \to \mathbf{C}^p$ be a holomorphic mapping in V such that $f(0) = 0$ and on each irreducible component of the analytic set $f^{-1}(0)$ in V, at least one (p, p) minor of the Jacobian matrix does not vanish identically. Then $\{f_1, \dots, f_p\}$ is a minimal defining system in V. Note that if $p = n$ and $f^{-1}(0) = \{0\}$, f is a minimal defining system if and only if $df(0) \neq 0$.

Let $\{f_1, \dots, f_p\}$ be a minimal defining system about the origin in $\mathbf{C}^n$. Since the set of singular points of $\{f_1 = f_2 = \dots = f_p = 0\} \cap V$ coincides exactly with the closed analytic subvariety

$$\mathcal{S} := \mathcal{S}ing\,(A) = \{z \in V,\ f_1 = \dots = f_p = 0,\ df = 0\}\,,$$

one can apply Hironaka's theorem and construct a resolution of singularities

$$\pi : \mathcal{X} \longrightarrow V ,$$

where π is proper, realizes a biholomorphism between $\mathcal{X} \setminus \pi^{-1}(\mathcal{S})$ and $V \setminus \mathcal{S}$, and is such that $\pi^{-1}(\mathcal{S})$ is an hypersurface with normal crossings. Since all $\pi^* f_j$ vanish in $\pi^{-1}(\mathcal{S})$, it follows from the Nullstellensatz that in any local chart on $\mathcal{X}$ one can write for every $j = 1, \dots, p$,

$$\pi^* f_j(w) = u_j(w) w_1^{\alpha_{j1}} \dots w_n^{\alpha_{jn}} , \tag{2.6}$$

where the α_{ji}, $j = 1, \dots, p$, $i = 1, \dots, n$ are positive integers and the u_j, $j = 1, \dots, n$, non vanishing holomorphic functions.
Let $F_j := \pi^* f_j$, $j = 1, \dots, p$. Our purpose here is to study the relation between the coherent sheaves $\mathcal{D}_{\mathcal{X}} F_1^{\lambda_1} \dots F_p^{\lambda_p}$ and $\mathcal{D}_V f_1^{\lambda_1} \dots f_p^{\lambda_p}$.
If we consider V as a complex n-manifold, let us define the two sheaves of modules

$$\Omega_V^{-1} = \mathrm{Hom}(\Omega_V , {}_n\mathcal{O}),$$

where Ω_V is the sheaf of holomorphic forms of degree n on V and

$$\mathcal{D}_{V \leftarrow \mathcal{X}} = \pi^{-1}(\mathcal{D}_V \otimes_{\mathcal{O}_V} \Omega_V^{-1}) \otimes \Omega_{\mathcal{X}},$$

where $\Omega_{\mathcal{X}}$ is the sheaf of holomorphic forms with degree n on $\mathcal{X}$. The above module $\mathcal{D}_{V \leftarrow \mathcal{X}}$ has the structure of $(\pi^{-1}\mathcal{D}_V, \mathcal{D}_{\mathcal{X}})$-bimodule. The integration of the coherent module $\mathcal{D}_{\mathcal{X}} F_1^{\lambda_1} \dots F_p^{\lambda_p}$ on $\mathcal{X}$ ([8], [14]) is defined to be the $\mathcal{D}_V$-module

$$\mathcal{R} = \int^0 \mathcal{D}_{\mathcal{X}} F_1^{\lambda_1} \dots F_p^{\lambda_p} = R^0 \phi_* (\mathcal{D}_{V \leftarrow \mathcal{X}} \otimes_{\mathcal{D}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}} F_1^{\lambda_1} \dots F_p^{\lambda_p})$$

where R^0 denotes the first derived functor of $\mathcal{D}_{V \leftarrow \mathcal{X}}$.
It follows from Theorem 4.2 [14] that the sheaf of left $\mathcal{D}_V$- modules $\mathcal{R}$ is a coherent sheaf of left $\mathcal{D}_V$-modules and is isomorphic to $\mathcal{D}_V f_1^{\lambda_1} \dots f_p^{\lambda_p}$ outside $\mathcal{S}$. Moreover, as was noted in [8, 14], the coherent sheaf of left $\mathcal{D}_V$-modules $\mathcal{R}$ has a global section u so that

$$\mathcal{D}_V u = \mathcal{D}_V[\lambda_1, \dots, \lambda_p] u \subset \int^0 \mathcal{D}_{\mathcal{X}} F_1^{\lambda_1} \dots F_p^{\lambda_p} . \tag{2.7}$$

The last relation is an equality on $V \setminus \mathcal{S}$ because $\pi : \mathcal{X} \setminus \pi^{-1}(\mathcal{S}) \to V \setminus \mathcal{S}$ is a biholomorphism. Let us describe the construction of the global section u as it is given in [8], p. 245. On V, we have the globally defined n-form $dz = dz_1 \wedge \ldots \wedge dz_n$. Its pulback $\pi^*(dz)$ is a globally defined n-form on manifold $\mathcal{X}$. By Proposition 2.12.6 in [8], p. 239, there exists a global section in $\int^0 i_{\mathcal{X}}(\mathcal{D}_{\mathcal{X}})$ denoted by $[\pi^*(dz)]$. Consider now the $\mathcal{D}_{\mathcal{X}}$-linear homomorphism $\eta : \mathcal{D}_{\mathcal{X}} \to \mathcal{D}_{\mathcal{X}} F_1^{\lambda_1} \ldots F_p^{\lambda_p}$ which is constructed by linear extension of the map $1_{\mathcal{X}} \to F_1^{\lambda_1} \ldots F_p^{\lambda_p}$. Since integration of modules corresponds to the action of a covariant functor, η induces a $\mathcal{D}_V$-linear sheaf homomorphism $\tilde{\eta}$ from $\int^0 \mathcal{D}_{\mathcal{X}}$ into $\mathcal{R}$. We define u as $u := \tilde{\eta}([\phi^*(dZ)])$. Under the minimal defining system condition, we have the following refined version of a result from [14, 8]

Lemma 2.1 *Let $\{f_1, \ldots, f_p\}$ be a minimal defining system in V. Then the coherent sheaf of $\mathcal{D}_V$- modules $\mathcal{R}/\mathcal{D}u$, where u has been constructed above, is equal to zero.*

Proof. Recall that $\mathcal{R} \cong \mathcal{D}u$ on $V \setminus \mathcal{S}$ (since π is a biholomorphism between $\mathcal{X} \setminus \pi^{-1}(\mathcal{S})$ and $V \setminus \mathcal{S}$.) On the other hand, $\mathcal{S}$ corresponds to the set of singular points of the set $V \cap f^{-1}(0)$ for which we constructed our resolution of singularities $\mathcal{X} \xrightarrow{\pi} V$. Our minimal defining system condition ensures that any point $z \in \mathcal{S}$ is a limit point of a sequence $\{z_n\}_n$ of regular points of $f^{-1}(0)$. This implies that for any point in $\mathcal{S}$, $\dim_z \mathcal{R}_z/(\mathcal{D}_V u)_z = 0$, where $\mathcal{R}_z$ and $(\mathcal{D}_V u)_z$ are sections of the corresponding sheaves at the point z, since for $z \notin \mathcal{S}$, the eqality $\mathcal{R} = \mathcal{D}_V u$ holds. Since every non-zero finitely generated $\mathcal{D}_V$-module has dimension bigger or equal to n, we get the desired result. $\diamondsuit$

We now continue with the introduction of p holomorphic parameters, $t_1, \ldots, t_p$, in order to deal first with what we will call the *quasi-homogeneous* case.

Lemma 2.2 *Let $\{f_1, \ldots, f_p\}$ be a minimal defining system in some open neighborhhood V of the origin in $\mathbf{C}^n$. Consider in $V \times \mathbf{C}^p$ (where coordinates are denoted as (z, t)) the holomorphic functions*

$$(z_1, \ldots, z_n, t_1, \ldots, t_p) \mapsto g_j(z, t) := t_j f_j(z)\,,\ j = 1, \ldots, p.$$

Then the system $(g_1, \ldots, g_p)$ is a minimal defining system in $V \times \mathbf{C}^p$.

Proof. Immediate by direct verification. ♢

Consider now the map

$$\phi := (\pi, \mathrm{Id}) : \ \mathcal{X} \times \mathbf{C}^p \longrightarrow V \times \mathbf{C}^p .$$

If we set $\mathcal{X}' := \mathcal{X} \times \mathbf{C}^p$ and $\mathcal{S}' := \mathcal{S} \times \mathbf{C}^p$, then ϕ induces a biholomorphism from $\mathcal{X}' \setminus \phi^{-1}(\mathcal{S}')$ into $V' \setminus \mathcal{S}'$. Let $G_j := \phi^* g_j$, $1 \le j \le p$, that is, in a local chart

$$G_j(w,t) = t_j u_j(w) w_1^{\alpha_{j1}} \cdots w_n^{\alpha_{jn}} \ , \ j = 1, \dots, p . \tag{2.8}$$

It follows from the quasi-homogeneous form of the g_j (and the G_j), due to the additional variables t_j, that the multiplication operators by $\lambda_1, \dots, \lambda_p$ induce $\mathcal{D}$-linear actions on the $\mathcal{D}_{V \times \mathbf{C}^p}$ (resp. $\mathcal{D}_{\mathcal{X}'}$) -sheaves of modules $\mathcal{D}_{V \times \mathbf{C}^p} g^{\underline{\lambda}}$ (resp. $\mathcal{D}_{\mathcal{X}'} G^{\underline{\lambda}}$.) Direct computations based on the simple expressions (2.8) for the G_j in local charts on $\mathcal{X}'$ show that we have the following

Lemma 2.3 *There exists a polynomial $b_G(\lambda_1, \dots, \lambda_p) \in \mathbf{C}[\lambda_1, \dots, \lambda_p]$, product of affine forms*

$$m_{L,0} + \sum_{j=1}^{p} m_{L,j} \lambda_j \ , \ L \in \mathcal{L}, \, m_{L,0} \in \mathbf{N}^*, \, (m_{L,1}, \dots, m_{L,p}) \in \mathbf{N}^p .$$

such that

$$b_G(\lambda_1, \dots, \lambda_p) G_1^{\lambda_1} \dots G_p^{\lambda_p} \in \mathcal{D}_{\mathcal{X}'} G_1^{\lambda_1 + 1} \cdots G_p^{\lambda_p + 1} . \tag{2.9}$$

If we look at the polynomial $b_G(\lambda_1, \dots, \lambda_p)$ as a sheaf homomorphism from the $\mathcal{D}_{\mathcal{X}'}$-module $\mathcal{D}_{\mathcal{X}'} G_1^{\lambda_1} \dots G_p^{\lambda_p}$ into $\mathcal{D}_{\mathcal{X}'} G_1^{\lambda_1+1} \dots G_p^{\lambda_p+1}$, then the question that arises naturally is what is its range. Let us describe it here. Let $\mathcal{O}_{\mathcal{X}'}$ be the sheaf of rings of germs of holomorphic functions on the manifold $\mathcal{X}'$. Consider also the sheaf of rings $\mathcal{O}_{\mathcal{X}'}[G_1^{-1}, \dots, G_p^{-1}]$ whose stalk at the point $x_0 \in \mathcal{X}'$ is

$$\mathcal{O}_{\mathcal{X}', x_0}[G_1^{-1}, \dots, G_p^{-1}] = \{ h G_1^{-v_1} \cdots G_p^{-v_p} | h \in \mathcal{O}_{\mathcal{X}', x_0}, \, v_j \in \mathbf{Z}, 1 \le j \le p \} .$$

Introducing new variables $(\lambda_1, \dots, \lambda_p) = \underline{\lambda}$, we consider also

$$\mathcal{O}_{\mathcal{X}'}[G_1^{-1}, \dots, G_p^{-1}, \underline{\lambda}] := \mathcal{O}_{\mathcal{X}'}[G_1^{-1}, \dots, G_p^{-1}][\lambda_1, \dots, \lambda_p] .$$

This is also a sheaf of rings on $\mathcal{X}'$ whose stalk at $x_0 \in \mathcal{X}$ is the ring of polynomials in $\underline{\lambda}$ with coefficients in $\mathcal{O}_{\mathcal{X}',x_0}[G_1^{-1},\ldots,G_p^{-1}]$. If $G^{\underline{\lambda}} := G_1^{\lambda_1}\ldots G_p^{\lambda_p}$, then the action of the differential operators ∂_l', $l=1,\ldots,n+p$ on $\mathcal{X}'$ (expressed in local coordinates (w,t)) on elements in $\mathcal{O}_{\mathcal{X}'}[G^{-1},\underline{\lambda}]G^{\underline{\lambda}}$ is defined as follows

$$\begin{aligned}\partial_l'(G_1^{-v_1}\ldots G_p^{-v_p}hG^{\underline{\lambda}}) &= \Big(\partial_l'h\prod_{i=1}^p G_i^{-v_i} - h\sum_{j=1}^p v_j\partial_l(G_j)G_j^{-v_j-1}\prod_{i\neq j}G_i^{-v_i}\\ &+ h\sum_{j=1}^p \lambda_j\partial_l'(G_j)G_j^{-1}\prod_{i=1}^p G_i^{-v_i}\Big)G^{\underline{\lambda}}. \qquad (2.10)\end{aligned}$$

This action induces a $\mathbf{C}[\lambda_1,\ldots,\lambda_p]$ linear mapping from $\mathcal{O}_{\mathcal{X}'}[G^{-1},\underline{\lambda}]G^{\underline{\lambda}}$ into itself; it induces on $\mathcal{O}_{\mathcal{X}'}[G^{-1},\underline{\lambda}]G^{\underline{\lambda}}$ a structure of $\mathcal{D}_{\mathcal{X}'}$ module. We can define also the action on $\mathcal{O}_{\mathcal{X}'}[G^{-1},\underline{\lambda}]G^{\underline{\lambda}}$ of the operator

$$\nabla:\ \mathcal{O}_{\mathcal{X}'}[G^{-1},\underline{\lambda}]G^{\underline{\lambda}} \longrightarrow \mathcal{O}_{\mathcal{X}'}[G^{-1},\underline{\lambda}]G^{\underline{\lambda}}$$

as follows

$$\nabla\Big(\big(\sum_{\underline{k}\in\mathbf{N}^p}\underline{\lambda}^{\underline{k}}\psi_{\underline{k}}\big)G^{\underline{\lambda}}\Big) = \Big(\sum_{\underline{k}\in\mathbf{N}^p}(\underline{\lambda}+\underline{1})^{\underline{k}}\psi_{\underline{k}}\Big)G_1\cdots G_pG^{\underline{\lambda}} \qquad (2.11)$$

(here $\underline{\lambda}^{\underline{k}} := \lambda_1^{k_1}\cdots\lambda_p^{k_p}$.) Since $\mathcal{D}_{\mathcal{X}'}G^{\underline{\lambda}}$ is a submodule of $\mathcal{O}_{\mathcal{X}'}[G^{-1},\underline{\lambda}]G^{\underline{\lambda}}$, we can conclude that

Lemma 2.4 *The mapping* $\nabla:\ \mathcal{D}_{\mathcal{X}'}G_1^{\lambda_1}\cdots G_p^{\lambda_p} \to \mathcal{D}_{\mathcal{X}'}G_1^{\lambda_1+1}\cdots G_p^{\lambda_p+1}$ *is* $\mathcal{D}_{\mathcal{X}'}$ *linear and injective.*

Since ∇ is $\mathcal{D}_{\mathcal{X}'}$-linear, it follows from (2.9) that $b_G(\lambda_1,\ldots,\lambda_p)G^{\underline{\lambda}} \in \nabla(\mathcal{D}_{\mathcal{X}'}G^{\underline{\lambda}})$. But ∇ is also injective, therefore there exists a $\mathcal{D}_{\mathcal{X}'}$-linear sheaf homomorphism ψ on $\mathcal{D}_{\mathcal{X}'}G^{\underline{\lambda}}$ such that $b_G(\lambda_1,\ldots,\lambda_p)=\nabla\psi$.

We recall here that the passage from $\mathcal{D}_{\mathcal{X}'}G^{\underline{\lambda}}$ to its direct sheaf image $\widetilde{\mathcal{R}}$ arises from a covariant functor from the category of sheaves of left $\mathcal{D}_{\mathcal{X}'}$-modules to the category of sheaves of $\mathcal{D}_{V\times\mathbf{C}^p}$-modules. Hence the sheaf homomorphisms ∇, ψ, $b_G(\lambda_1,\ldots,\lambda_p)$ induce $\mathcal{D}_{V\times\mathbf{C}^p}$-linear sheaf homomorphisms on $\widetilde{\mathcal{R}}=\mathcal{D}\tilde{u}$ (the existence of $\tilde{u}$ follows from Lemma 2.1 and Lemma 2.2, we consider just the minimal defining system g instead of f.) Therefore

$$b_G(\lambda_1,\ldots,\lambda_p)\widetilde{\mathcal{R}} = (\nabla\psi)\widetilde{\mathcal{R}} = \nabla(\psi\widetilde{\mathcal{R}}) \subset \nabla\widetilde{\mathcal{R}} = \nabla\mathcal{D}\tilde{u}\,. \qquad (2.12)$$

We claim now that there exists a $\mathcal{D}_{V\times\mathbf{C}^p}$-linear sheaf homomorphism from $\mathcal{D}_{V\times\mathbf{C}^p}\tilde{u}$ onto $\mathcal{D}_{V\times\mathbf{C}^p}g_1^{\lambda_1}\dots g_p^{\lambda_p}$: just define a map that takes $\tilde{u}$ to $g_1^{\lambda_1}\dots g_p^{\lambda_p}$ and then extend linearly. This map has the desired property ([8], p.246.) Therefore, combining the above assertions, we get $b_G(\lambda_1,\dots,\lambda_p)\widetilde{\mathcal{R}}\subset\mathcal{D}_{V\times\mathbf{C}^p}\tilde{u}$ and hence by the above epimorphism, we conclude that, as germs at the origin

$$b_G(\lambda_1,\dots,\lambda_p)g_1^{\lambda_1}\cdots g_p^{\lambda_p}\in\nabla(\mathcal{D}_{\mathbf{C}^{n+p},0}g_1^{\lambda_1}\dots g_p^{\lambda_p})=\mathcal{D}_{\mathbf{C}^{n+p},0}g_1^{\lambda_1+1}\dots g_p^{\lambda_p+1}.$$

Hence we have proved the following form of the Bernstein-Sato relations

Proposition 2.3 *Let $\{f_1,\dots,f_p\}$ be a minimal defining system in V. Define in $V\times\mathbf{C}^p$ the system $(g_1,\dots,g_p)$, where $g_j(z,t):=t_jf_j(z)$, $j=1,\dots,p$. Then there exists an operator $\mathcal{Q}(z,t,\partial_z,\partial_t)\in\mathcal{D}_{\mathbf{C}^{n+p},0}$ and a polynomial $b_g=b_G$ in $\mathbf{C}[\lambda_1,\dots,\lambda_p]$, which is a product of affine forms*

$$m_{L,0}+\sum_{j=1}^{p}m_{L,j}\lambda_j\ ,\ \ m_{L,0}\in\mathbf{N}^*,m_{L,j}\in\mathbf{N}\,,$$

such that

$$b_g(\lambda_1,\dots,\lambda_p)g_1^{\lambda_1}\cdots g_p^{\lambda_p}=\mathcal{Q}(z,t,\partial_z,\partial_t)g_1^{\lambda_1+1}\cdots g_p^{\lambda_p+1}\,,$$

the identity being understood in terms of germs at the origin.

Repeating verbatim the argument in [8] we deduce

Proposition 2.4 *Let $(f_1,\dots,f_p)$ be a minimal defining system about the origin in $\mathbf{C}^n$. Then there exists a neighborhood ω of the origin, a polynomial*

$$B(\underline{\lambda})=\prod_{L\in\mathcal{L}}(m_{L,0}+\sum_{j=1}^{p}m_{L,j}\lambda_j)$$

where $m_{L,0}\in\mathbf{N}^*$, $m_{L,1},\dots,m_{L,p}\in\mathbf{N}$, such that

$$B(\underline{\lambda})f_1^{\lambda_1}\dots f_p^{\lambda_p}\in\mathcal{D}_V[\lambda_1,\dots,\lambda_p]f_1^{\lambda_1+1}\cdots f_p^{\lambda_p+1}$$

3 About Kashiwara's functional equations

Let us recall that if f is a function of n-variables holomorphic in a neighborhood V of the origin, such that $f(0) = 0$ and $df = 0$ implies $f = 0$, then the $\mathcal{D}_V[\lambda]$-module $\mathcal{D}_V[\lambda]f^\lambda$ is a coherent $\mathcal{D}_V$-module. From this, it follows, if $V_0 \subset\subset V$, that for some $q \in \mathbf{N}$,

$$\mathcal{D}_{V_0}[\lambda]f^\lambda \subset \sum_{k=0}^{q} \lambda^k \mathcal{D}_{V_0} f^\lambda.$$

Therefore one can find a functional equation of the form

$$\Big(\lambda^{q+1} - \sum_{k=0}^{q} \lambda^k \mathcal{Q}_k(z, \partial)\Big) f^\lambda = 0, \tag{3.1}$$

where the operators $\mathcal{Q}_k$, $k = 0, \dots, q$ are global sections of $\mathcal{D}_{V_0}$, that is we can find an operator of the form (1.8) with $M = q + 1$ that annihilates f^λ. We will use the following immediate extension of this result

Proposition 3.1 *Let $f_1, \dots, f_p$ be p holomorphic functions in some neighborhood of the origin, such that the $\mathcal{D}_V[\lambda_1, \dots, \lambda_p]$-module*

$$\mathcal{D}_V[\lambda_1, \dots, \lambda_p] f_1^{\lambda_1} \dots f_p^{\lambda_p}$$

is a coherent $\mathcal{D}_V$-module. Then, given any $V_0 \subset\subset V$, there are p operators of the form

$$\lambda_j^M - \sum_{\substack{\underline{k} \in \mathbf{N}^p \\ k_1 + \dots + k_p \le M-1}} \lambda_1^{k_1} \cdots \lambda_p^{k_p} \mathcal{Q}_{j,\underline{k}}(z, \partial)\,, \ j = 1, \dots, p \tag{3.2}$$

(where the $\mathcal{Q}_{j,\underline{k}}$ are global sections of $\mathcal{D}_{V_0}$) which annihilate $f_1^{\lambda_1} \dots f_p^{\lambda_p}$ on V_0.

Proof. Multiplications by $\lambda_1, \dots, \lambda_p$ act as a $\mathcal{D}_V$ -linear operators on the module $\mathcal{D}_V[\underline{\lambda}] f_1^{\lambda_1} \dots f_p^{\lambda_p}$. Hence, it follows from the coherence that, given $V_0 \subset\subset V$, there exists some integer $q \in \mathbf{N}$ such that

$$\mathcal{D}_V[\lambda_1, \dots, \lambda_p] f_1^{\lambda_1} \dots f_p^{\lambda_p} \subset \sum_{\substack{\underline{k} \in \mathbf{N}^p \\ k_1 + \dots + k_p \le q}} \underline{\lambda}^{\underline{k}} \mathcal{D}_V f_1^{\lambda_1} \dots f_p^{\lambda_p}.$$

Therefore, we have in particular, for any $j \in \{1, \ldots, p\}$,

$$\lambda_j^{q+1} f_1^{\lambda_1} \ldots f_p^{\lambda_p} \in \sum_{\substack{\underline{k} \in \mathbf{N}^p \\ k_1 + \ldots + k_p \leq q}} \underline{\lambda}^{\underline{k}} \mathcal{D}_V f_1^{\lambda_1} \ldots f_p^{\lambda_p},$$

This provides us with the set of operators we are looking for (take $M = q+1$) and concludes the proof of the proposition. $\diamondsuit$

In the case $p = 1$, assuming $f(0) = 0$ and that in V, $df = 0$ implies $f = 0$, the algebraic dependency of f over its jacobian ideal implies [14] a much more precise result; in fact, in this case, the annihilator of f^λ on V contains an operator of the form

$$\begin{aligned} \lambda^M - \sum_{k=1}^{M} \lambda^{M-k} \mathcal{Q}_k(z, \partial_z), \\ deg_\partial \mathcal{Q}_k \leq k,\ k = 1, \ldots, M, \end{aligned} \tag{3.3}$$

where $\mathcal{Q}_k(z, \partial) \in \mathcal{D}_V$. Such a result relies on the description of the characteristic variety of the $\mathcal{D}_{V \times \mathbf{C}}$-module $\mathcal{D}_{V \times \mathbf{C}}(tf)^\lambda$, where t is an additional variable [2, 8, 14]. In [7], H. Biosca and H. Meynadier have extended this result of M. Kashiwara (the existence of operators of the form (3.3) in the annihilator of f^λ) to the case $p > 1$, when $f_1, \ldots, f_p$ define a complete intersection in a neighborhood V of the origin in $\mathbf{C}^n$. Their result relies on the description of the two characteristic varieties W_f (resp. $W_f^\#$) of $\mathcal{D}_V f_1^{\lambda_1} \ldots f_p^{\lambda_p}$, considered as a $\mathcal{D}_V$-module, (resp. of $\mathcal{D}_V[\lambda_1, \ldots, \lambda_p] f_1^{\lambda_1} \ldots f_p^{\lambda_p}$, considered as a $\mathcal{D}_V[\lambda_1, \ldots, \lambda_p]$ module.) Namely

$$\begin{aligned} W_f &= \overline{\{(z, \sum_{j=1}^{p} \lambda_j df_j,\ z \in V,\ df \neq 0,\ \underline{\lambda} \in \mathbf{C}^p\}} \\ W_f^\# &= \overline{\{(z, \sum_{j=1}^{p} \lambda_j df_j,\ \lambda_1 f_1(z), \ldots, \lambda_p f_p(z)),\ z \in V,\ df \neq 0,\ \underline{\lambda} \in \mathbf{C}^p\}}. \end{aligned}$$

The finiteness of the projection map

$$\Pi :\ W_f^\# \longrightarrow W_f \tag{3.4}$$

implies that the stalk $\mathcal{D}_{\mathbf{C}^n,0}[\lambda_1, \ldots, \lambda_p] f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ is of finite type as a $\mathcal{D}_{\mathbf{C}^n,0}$-module, which is enough to ensure the existence of a set of operators of the form (3.2), everything being understood at the level of stalks at the origin. In fact, the finiteness of this projection map implies much more, as it appears in the following result from [7]

Proposition 3.2 *Let $(f_1, \ldots, f_p)$ define a germ of complete intersection at the origin in $\mathbf{C}^n$. The projection map Π from $W_f^{\#}$ into W_f is a finite morphism if and only if, for any $j = 1, \ldots, p$, the annihilator of $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ contains an operator of the form*

$$\lambda_j^{M_j} - \sum_{k=1}^{M_j} \mathcal{Q}_{j,k}(z, \partial_z, \underline{\lambda}) \lambda_j^{M_j - k}$$

where the $\mathcal{Q}_{j,k}$, $j = 1, \ldots, p$, $k = 1, \ldots, M_j$, are elements in $\mathcal{D}_{\mathbf{C}^n,0}[\underline{\lambda}]$ such that $\deg_{\partial,\underline{\lambda}} \mathcal{Q}_{j,k} \leq k$ for any $j \in \{1, \ldots, p\}$, $k = 1, \ldots, M_j$, and the homogeneous part of degree k in $\mathcal{Q}_{j,k}$ being $\underline{\lambda}$-free.

Let us give the following example (found in [7])

Example 3.1. For the mapping

$$\begin{array}{rcl} f:\ \mathbf{C}^3 & \longrightarrow & \mathbf{C}^2 \\ (z_1, z_2, z_3) & \longrightarrow & (z_1^2 - z_2^2 z_3, z_2)\,, \end{array}$$

one can check here the finiteness of the projection morphism Π.

Remark 3.1. The finiteness of the projection morphism Π, as noticed in [7], implies that the germ of the set of critical points is necesseraly included in the hypersurface $f_1 \ldots f_p = 0$ (which means that $f_1 \ldots f_p$ lies in the radical of the Jacobian ideal.) For example

$$(z_1, z_2) \longrightarrow (z_1, z_1^2 + z_2^2)$$

fails to satisfy these requirements. The finiteness of the morphism Π appears as a sufficient condition for the coherence of the sheaf $\mathcal{D}_V[\lambda_1, \ldots, \lambda_p] f^{\underline{\lambda}}$ (for some convenient neighborhood V of the origin) as a $\mathcal{D}_V$-module. Nevertheless the condition is certainly too strong.

4 Some positive results on the existence of the unrestricted limit (1.5)

In this section $f_1, \dots f_p$ are p holomorphic functions defining a complete intersection in a neighborhood V of the origin in $\mathbf{C}^n$.

Theorem 4.1 *Assume that the $\mathcal{D}_V[\lambda_1, \lambda_2]$-module $\mathcal{D}_V[\lambda_1, \lambda_2] f_1^{\lambda_1} f_2^{\lambda_2}$ is a coherent $\mathcal{D}_V$-module. Then the unrestricted limit*

$$\lim_{\underline{\epsilon} \mapsto \underline{0}} \int_{\substack{|f_1(z)|=\epsilon_1 \\ |f_2(z)|=\epsilon_2}} \frac{\varphi}{f_1 f_2} \tag{4.1}$$

exists for any $\varphi \in \mathcal{D}^{n,n-p}(V)$.

Proof. Consider the Mellin Transform of

$$(\epsilon_1, \epsilon_2) \longrightarrow I(\underline{\epsilon}; \varphi) = \frac{1}{(2\pi i)^2} \int_{\substack{|f_1(\zeta)|=\epsilon_1 \\ |f_2(\zeta)|=\epsilon_2}} \frac{\varphi}{f_1 f_2}$$

This is exactly (for $\Re\lambda_1 >> 1, \Re\lambda_2 >> 1$) the function

$$\underline{\lambda} \longrightarrow J(\underline{\lambda}; \varphi) = \frac{\lambda_1 \lambda_2}{4\pi^2} \int_V |f_1|^{2(\lambda_1 - 1)} |f_2|^{2(\lambda_2 - 1)} \overline{\partial f_1} \wedge \overline{\partial f_2} \wedge \varphi.$$

We know that because of the existence of the set of equations of the form (2.1) and of Proposition 2.2, the function $\underline{\lambda} \longrightarrow J(\underline{\lambda}; \varphi)$ can be continued as a meromorphic function in the whole of $\mathbf{C}^2$, the polar set being a union of hyperplanes of the form

$$m_{L,0} + m_{L,1}(\lambda_1 + k) + m_{L,2}(\lambda_2 + k) = 0, \; k \in \mathbf{N}$$
$$m_{L,0} \in \mathbf{N}^*, \; m_{L,1}, m_{L,2} \in \mathbf{N}, \text{ for any } L \in \mathcal{L},$$

where $\mathcal{L}$ is a finite set as in Proposition 2.2. Denote by $\underline{\lambda} \mapsto J(\underline{\lambda}; \varphi)$ this meromorphic continuation. It follows from Proposition 3.2 that for any (γ_1, γ_2) such that

$$m_{L,0} + m_{L,1}(\gamma_1 + k) + m_{L,2}(\gamma_2 + k) \neq 0,$$

for any $L \in \mathcal{L}$ and any $k \in \mathbf{N}$, the function

$$(y_1, y_2) \longrightarrow J(\gamma_1 + iy_1, \gamma_2 + iy_2; \varphi)$$

is in the space $\mathcal{S}(\mathbf{R}^2)$ of rapidly decreasing smooth functions. By Mellin formula, we get for $\epsilon_1 > 0, \epsilon_2 > 0$

$$I(\underline{\epsilon}; \varphi) = \frac{1}{(2\pi i)^2} \int\limits_{\gamma_1^0 + i\mathbf{R}} \int\limits_{\gamma_2^0 + i\mathbf{R}} \frac{J(\underline{\lambda}, \varphi)}{\lambda_1 \lambda_2} \epsilon_1^{-\lambda_1} \epsilon_2^{-\lambda_2} d\lambda_1 d\lambda_2,$$

where γ_1^0, γ_2^0 are strictly positive numbers which are chosen large enough. Moving γ_1, γ_2 towards the origin (this we can do because of the Cauchy formula), using the uniform rapid decrease of

$$(y_1, y_2) \longrightarrow J(\gamma_1 + iy_1, \gamma_2 + iy_2; \varphi),$$

when $\gamma_1, \gamma_2) \in [-\delta_1, \gamma_1^0] \times [-\delta_2, \gamma_2^0]$ and the fact that all $m_{L,0}$ are strictly positive we get that for δ_1, δ_2 small enough

$$I(\underline{\epsilon}; \varphi) = \frac{1}{(2\pi i)^2} \int\limits_{\gamma_1^0 + i\mathbf{R}} \int\limits_{-\delta_2 + i\mathbf{R}} J(\underline{\lambda}; \varphi) \epsilon_1^{-\lambda_1} \epsilon_2^{-\lambda_2} \frac{d\lambda_1 d\lambda_2}{\lambda_1 \lambda_2} +$$

$$+ \frac{1}{2\pi i} \int\limits_{\gamma_1^0 + i\mathbf{R}} J(\lambda_1, 0; \varphi) \epsilon^{-\lambda_1} \frac{d\lambda_1}{\lambda_1} =$$

$$= \frac{1}{(2i\pi)^2} \int_{-\delta_1 + i\mathbf{R}} \int_{-\delta_2 + i\mathbf{R}} J(\underline{\lambda}; \varphi) \epsilon_1^{-\lambda_1} \epsilon_2^{-\lambda_2} \frac{d\lambda_1 d\lambda_2}{\lambda_1 \lambda_2} +$$

$$+ \frac{1}{2\pi i} \Big(\int\limits_{-\delta_1 + i\mathbf{R}} J(\lambda_1, 0; \varphi) \epsilon_1^{-\lambda_1} \frac{d\lambda_1}{\lambda_1} - \int_{-\delta_2 + i\mathbf{R}} J(0, \lambda_2; \varphi) \epsilon_2^{-\lambda_2} \frac{d\lambda_2}{\lambda_2} \Big) + J(\underline{0}; \varphi)\,. \tag{4.2}$$

Since the function

$$(\epsilon_1, \epsilon_2) \longrightarrow \int\limits_{-\delta_1 + i\mathbf{R}} \int\limits_{-\delta_2 + i\mathbf{R}} J(\underline{\lambda}, \varphi) \epsilon_1^{-\lambda_1} \epsilon_2^{-\lambda_2} \frac{d\lambda_1 d\lambda_2}{\lambda_1 \lambda_2}$$

can be estimated by $C\epsilon_1^{\delta_1}\epsilon_2^{\delta_2}$, due to the rapid decrease of $\underline{\lambda} \longrightarrow J(\underline{\lambda};\varphi)$ on the line $\lambda_1 = -\delta_1 + i\mathbf{R}$, $\lambda_2 = -\delta_2 + i\mathbf{R}$, and the functions

$$\epsilon_1 \mapsto \int_{-\delta_1+i\mathbf{R}} J(\lambda_1, 0;\varphi)\epsilon_1^{-\lambda_1}\frac{d\lambda_1}{\lambda_1}$$

$$\epsilon_2 \mapsto \int_{-\delta_2+i\mathbf{R}} J(0, \lambda_2;\varphi)\epsilon_2^{-\lambda_2}\frac{d\lambda_2}{\lambda_2}$$

are estimated respectively by $C\epsilon_1^{\delta_1}$ and $C\epsilon_2^{\delta_2}$ for similar reasons, we get that

$$\begin{aligned}\lim_{\underline{\epsilon}\mapsto\underline{0}} I(\epsilon_1,\epsilon_2;\varphi) &= \lim_{\underline{\epsilon}\mapsto\underline{0}} <\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}, J(\underline{\lambda},\varphi)\epsilon_1^{\lambda_1}\epsilon_2^{-\lambda_2}>_0 \\ &= J(\underline{0};\varphi) =<\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2},\varphi> .\end{aligned}$$

This ends the proof of Theorem 4.1. $\diamondsuit$

Example 4.1. An important example where we know that the stalk of the sheaf at the origin $\mathcal{D}_{\mathbf{C}^n,0}[\lambda_1,\dots,\lambda_p]f_1^{\lambda_1}\dots f_p^{\lambda_p}$ is of finite type over $\mathcal{D}_{\mathbf{C}^n,0}$ (and therefore we can apply the previous result when V is a sufficiently small neighborhood of the origin) corresponds to the case when the projection map

$$\Pi:\ W_f^{\#} \longrightarrow W_f$$

introduced in (3.4) is finite (see [7], section 3.) We can therefore state the following

Corollary 4.1 *Let (f_1, f_2) two elements in ${}_n\mathcal{O}$ which define a germ of complete intersection. Assume that the projection map $W_f^{\#} \stackrel{\Pi}{\longrightarrow} W_f$ introduced in (3.5) satisfies $\Pi^{-1}(\underline{0}) = \{\underline{0}\}$. Then there exists a neighborhood V of the origin such that, for any $\varphi \in \mathcal{D}^{n,n-2}(V)$,*

$$\lim_{\substack{\epsilon_1\mapsto 0\\ \epsilon_2\mapsto 0}} \frac{1}{(2\pi i)^2}\int_{\substack{|f_1(\zeta)|=\epsilon_1\\ |f_2(\zeta)|=\epsilon_2}} \frac{\varphi}{f_1 f_2} =<\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2},\varphi> \tag{4.3}$$

Example 4.2. For example, if $n = 3$ and $m \in \mathbf{N}^*$, we have, for any φ in $\mathcal{D}^{3,1}(V)$, where V is a sufficiently small neighborhood of the origin in $\mathbf{C}^3$

$$\lim_{\substack{\epsilon_1\mapsto 0\\ \epsilon_2\mapsto 0}} \frac{1}{(2\pi i)^2}\int_{\substack{|\zeta_1^2-\zeta_2^2\zeta_3|=\epsilon_1\\ |\zeta_2^m|=\epsilon_2}} \frac{\varphi}{\zeta_2^m(\zeta_1^2-\zeta_2^2\zeta_3)} =<\bar\partial\frac{1}{(\zeta_1^2-\zeta_2^2\zeta_3)}\wedge\bar\partial\frac{1}{\zeta_2^m},\varphi> .$$

Remark 4.1. In Bjork's example (1.7) where $f_1(z_1, z_2) = z_1,\ f_2(z_1, z_2) = z_2^3 + z_1 + z_1^2$, since we know from [9], sec.7.2, that the unrestricted limit does not exist, we are sure that the stalk $\mathcal{D}_{\mathbf{C}^2,0}[\lambda_1, \lambda_2] z_1^{\lambda_1}(z_2^3 + z_1 + z_1^2)^{\lambda_2}$ is not of finite type as a $\mathcal{D}_{\mathbf{C}^2,0}$-module. In fact, in the codimension 2 case, any negative example for the unrestricted continuity of (1.5) provides an example of non-coherence for the sheaf $\mathcal{D}_V[\lambda_1, \lambda_2] f_1^{\lambda_1} f_2^{\lambda_2}$ as a $\mathcal{D}_V$-module.

Example 4.3. Corollary 4.1 holds if the germ (f_1, f_2) satisfies the Sabbah-Loeser conditions

$$df_1 \wedge df_2 = 0 \implies f_1 \cdot f_2 = 0 \tag{4.4}$$

$$(f_1, f_2) \text{ has no blowing up in codimension } 0 \tag{4.5}$$

For example these conditions are fulfilled if (f_1, f_2) define a complete intersection with isolated singularity, with the additional constraint

$$df_1 \wedge df_2 = 0 \implies f_1 \cdot f_2 = 0.$$

Note that the unrestricted limit (1.5) may exist even if the coherence condition is not fulfilled. In this direction we have already mentionned the example of J. E. Björk in [9] where f_1, f_2 are homogeneous polynomials. When $f_1(z_1, z_2) = z_1,\ f_2(z) = z_1^2 + z_2^2$ (these are homogeneous, so that the unrestricted limit (1.5) exists for any test form in $\mathcal{D}^{2,0}(\mathbf{C}^2)$), one can show that there are test forms in $\mathcal{D}^{(n,n-2)}(V)$, where V is any arbitrary neighborhhood of the origin, for which the rapid decrease of the function

$$\underline{\lambda} \mapsto J(\underline{\lambda}; \varphi)$$

cannot be realized (this can be seen using the proper map $\pi :\ \mathcal{X} \mapsto V$, where $\mathcal{X}$ is the toric variety corresponding to the convex hull of $\{(0,1) + [0, \infty[^2\} \cup \{(1,0) + [0, \infty[^2\}$.) For such an example, the coherence condition in Proposition 3.1 is certainly not fulfilled.

5 About asymptotic developments

Let us recall the results relative to the case $p = 1$. Classical inversion theorems about the Mellin Transform show that, since

$$\lambda \longrightarrow |f|^\lambda$$

has a meromorphic continuation which is rapidly decreasing on vertical lines $\gamma + i\mathbf{R}$, then

$$\epsilon \longrightarrow \frac{1}{2\pi i} \int\limits_{|f|=\epsilon} \frac{\varphi}{f}$$

admits an analytic development near the origin in the basis $(1, \epsilon^{\alpha}(\log \epsilon)^{\beta})$, $\alpha \in \mathbf{Q}^{+*}, \beta \in \mathbf{N}$. When $p > 1$, and $f_1, \ldots, f_p$ define a complete intersection in a neighborhood V of the origin, we have under the hypothesis of Theorem 3.1, a similar condition with respect to the rapid decrease on vertical lines $\underline{\gamma} + i\mathbf{R}^p$ for the meromorphic continuation of the multivariable Mellin Transform of the function $\underline{\epsilon} \mapsto I(\underline{\epsilon}; \varphi)$, when $\varphi \in \mathcal{D}^{n,n-p}(V)$. Unfortunately, even in the case $p = 2$, there remain considerable difficulties (see for example [3]) in order to deduce from such a behavior some asymptotic developments for the

$$(\epsilon_1, \epsilon_2) \longrightarrow I(\underline{\epsilon}, \varphi)$$

in terms of $(\epsilon_1^{\alpha_1}\epsilon_2^{\alpha_2}(\log \epsilon_1)^{\beta_1}(\log \epsilon_2)^{\beta_2})$, $\alpha_1, \alpha_2 \in \mathbf{Q}$, $\beta_1, \beta_2 \in \mathbf{N}$. Trying to avoid these difficulties, we attempted to study *one parameter* asymptotic approximations to the residual currents associated to p functions. There are two of them which are interesting (see [20]).

$$< \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}, \varphi >= \lim_{\epsilon \mapsto 0} \frac{c_p}{\epsilon^p} \int\limits_{\{\|f\|^2=\epsilon\}\cap V} \sum_1^p (-1)^{k-1} \bar{f}_k \overline{df_k} \wedge \varphi \qquad (5.1)$$

$$< \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_p}, \varphi >= \lim_{\epsilon \mapsto 0} p c_p \tau \int \frac{\overline{\partial f_1} \wedge \cdots \wedge \overline{\partial f_p} \wedge \varphi}{(\|f\|^2 + \tau)^{p+1}} , \qquad (5.2)$$

where

$$c_p := \frac{(-1)^{\frac{p(p-1)}{2}}(p-1)!}{(2\pi i)^p}.$$

As for the approach (5.1), we are reduced to classical problems in one variable, since the one dimensional Mellin transform of

$$\epsilon \longrightarrow \frac{c_p}{\epsilon^p} \int\limits_{\{\|f\|^2=\epsilon\}\cap V} \sum_{k=1}^p (-1)^{k-1} \bar{f}_k \overline{df_k} \wedge \varphi$$

is

$$\lambda \longrightarrow pc_p \int \|f\|^{2(\lambda+1-p)} \overline{\partial f_1} \wedge \dots \overline{\partial f_p} \wedge \varphi. \tag{5.3}$$

The meromorphic function (5.3) has its poles in $\{\gamma \in \mathbf{Q},\ \gamma \leq -1\}$; the pole at -1 is simple and the value of the residue at -1 is

$$< \bar{\partial}\frac{1}{f_1} \wedge \dots \wedge \bar{\partial}\frac{1}{f_p}, \varphi >$$

(see [20].) We get here, since there exists an operator

$$\lambda^M - \sum_{k=1}^{M-1} \mathcal{Q}_k(z, \bar{z}, \partial, \bar{\partial})\lambda^k,$$

with $\mathcal{C}^\infty$ coefficients that annihilates $\|f\|^{2\lambda}$, the rapid decrease on the vertical lines for the function (5.3), and therefore , using the classical techniques developed by Jeanquartier, Barlet and Maire [11, 1, 2], we get the asymptotic development for (5.1) (as a function of ϵ) in terms of the basis $(1, \epsilon^\alpha (\log \epsilon)^\beta)$, $\alpha \in \mathbf{Q}^{+*}$, $\beta \in \mathbf{N}$. More interesting from our point of view is the second approach (5.2) where the two dimensional Mellin Transform plays an important intermediate role, even though we know also in this case (by a similar one variable argument) the existence of an asymptotic development.

Proposition 5.1 *Let $f_1, \dots, f_p$ define a complete intersection in a neighborhood V of the origin in $\mathbf{C}^n$. Then, for any test form $\varphi \in \mathcal{D}^{(n,n-p)}(V)$, the map*

$$\tau \mapsto \frac{(-1)^{p(p-1)/2} p! \tau}{(2i\pi)^p} \int_V \frac{\overline{\partial f_1} \wedge \dots \wedge \overline{\partial f_p} \wedge \varphi}{(\|f\|^2 + \tau)^{p+1}}$$

is continuous at the origin, takes the value

$$< \bar{\partial}\frac{1}{f_1} \wedge \dots \wedge \bar{\partial}\frac{1}{f_p}, \varphi >$$

at $\tau = 0$ and admits an asymptotic development in the basis $(1, \tau^\alpha (\log \tau)^\beta)$, $\alpha \in \mathbf{Q}^{+*}$, $\beta \in \mathbf{N}$ about the origin. Moreover, if $\mathcal{D}_V[\underline{\lambda}]f^\lambda$ is coherent as a $\mathcal{D}_V$-sheaf of modules, then the coefficients in this development can be computed *in terms of sums of Leray iterated residues at points in* $\bigcup_{j=1}^{p}\{\Re\lambda_j \le -1\}$ *for the function*

$$\underline{\lambda} \mapsto \frac{(-1)^{p(p-1)/2}\Gamma(|\underline{\lambda}|+p+1)\prod_{j=1}^{p}\Gamma(-\lambda_j)J(\underline{\lambda}+\underline{1};\varphi)\tau^{-|\underline{\lambda}|-p}}{(2i\pi)^p\prod_{j=1}^{p}(\lambda_j+1)}$$

($|\underline{\lambda}| := \lambda_1+\cdots+\lambda_p$) *along collections of* p *hyperplanes (with independent directions) either of the form* $\lambda_j = q-1$, $q\in\mathbf{N}$, $j\in\{1,\dots,p\}$, $|\lambda| = -p-1-q$, $q\in\mathbf{N}$, *or*

$$m_{L,0}+\sum_{j=1}^{p} m_{L,j}(\lambda_j+q) = 0\,,\ \ q\in\mathbf{N}$$

where

$$m_{L,0}+\sum_{j=1}^{p} m_{L,j}\lambda_j$$

divides a Bernstein-Sato polynomial for $f^{\underline{\lambda}}$.

Proof. The existence of an asymptotic development is a standard thing; it can be achieved under the sole hypothesis that $(f_1,\dots,f_p)$ define a complete intersection in V. For any ζ such that $\|f(\zeta)\|^2\neq 0$, we may use the classical formula: for any $\tau>0$

$$p!\frac{\tau}{(\|f(\zeta)\|^2+\tau)^{p+1}} = \frac{1}{2i\pi}\int_{-\gamma+i\mathbf{R}}\Gamma(-s)\Gamma(p+1+s)\|f(\zeta)\|^{2s}\tau^{-p-s}ds\,,$$

where $0<\gamma<p$ (see [5]). Let now $\varphi\in\mathcal{D}^{n,n-p}(V)$. When γ is sufficiently small, one can prove, using a resolution of singularities as in [4] , that

$$\int\int_{\mathbf{V}\times\{-\gamma+i\mathbf{R}\}}|\Gamma(-s)||\Gamma(p+1+s)||\|f(\zeta)\|^{-2\gamma}\|\overline{\partial f_1}\wedge\cdots\wedge\overline{\partial f_p}\wedge\varphi\||ds|<\infty\,.$$

It follows from Fubini's theorem that

$$\frac{p!(-1)^{p(p-1)/2}\tau}{(2i\pi)^p}\int_V\frac{\overline{\partial f_1}\wedge\cdots\wedge\overline{\partial f_p}\wedge\varphi}{(\|f(\zeta\|^2+\tau)^{p+1}} =$$

$$= \frac{1}{(2i\pi)}\int_{-\gamma+i\mathbf{R}}\Gamma(-s)\Gamma(p+1+s)F(\underline{\lambda};\varphi)\tau^{-p-s}ds \tag{5.4}$$

where

$$F(\underline{\lambda};\varphi) := \frac{(-1)^{p(p-1)/2}}{(2i\pi)^p}\int_V \|f\|^{2s}\overline{\partial f_1}\wedge\cdots\wedge\overline{\partial f_p}\wedge\varphi\,.$$

We also know from [20] that the function

$$\mu\in\mathbf{C}\mapsto F(\mu;\varphi)$$

(defined for $\Re\mu>0$) admits a meromorphic continuation $u\mapsto F(u;\varphi)$ to the whole complex plane, with poles in $\mathbf{Q}\cap\,]-\infty,-p]$; moreover (see also [20]), this analytic continuation satisfies uniform rapid decrease estimates at infinity in any vertical strip $[\alpha,\beta]+i\mathbf{R}$ which is free of poles. The pole at $-p$ is a simple one and the residue at this point equals

$$<\overline{\partial}\frac{1}{f_1}\wedge\cdots\wedge\overline{\partial}\frac{1}{f_p}\,,\,\varphi>\ .$$

The poles of the function

$$s\mapsto\Gamma(-s)\Gamma(p+1+s)$$

which lie in the half plane $\Re s<-\gamma$ are $-p-1,-p-2,\ldots$. If we apply the uniform boundedness of $\mu\mapsto F(\mu;\varphi)$ on vertical strips in the complex plane which are pole free for this function, we deduce, moving the line integral in the right hand side of (5.4) step by step to the left, the existence of an asymptotic development for (5.4) (as a function of τ) with respect to the basis $(1,\tau^\alpha(\log\tau)^\beta)$, $\tau\in\mathbf{Q}$, $\alpha>0$, $\beta\in\mathbf{N}$.

The interesting additional thing here is the relation between this asymptotic development and the description of the polar set of

$$(\lambda_1,\ldots,\lambda_p)\mapsto J(\underline{\lambda};\varphi)$$

introduced in (1.9); such a polar set $\mathcal{S}ing\,(J)$ is (see [4], Proposition 3.6) included in a collection of hyperplanes with equations

$$m_{L,0}+\sum_{j=1}^p m_{L,j}(\lambda_j+k-1)=0\,,\ k\in\mathbf{N}$$

where the vectors

$$(m_{L,0},\ldots,m_{L,p})\in\mathbf{N}^*\times(\mathbf{N}^p)^*$$

are indexed by a finite set $\mathcal{L}$. If we assume the coherence assumption, which we will do from now on, we know that the function

$$(\lambda_1, \dots, \lambda_p) \mapsto J(\underline{\lambda}; \varphi)$$

is uniformly rapidly decreasing (in the imaginary direction) in any *vertical* strip $K + i\mathbf{R}^p$ (where K is a compact subset in $\mathbf{R}^p$) such that K does not intersect $\mathcal{S}ing\,(J) \cap \mathbf{R}$. For any ζ in V and any $\tau > 0$ such that $f_1 \cdots f_p(\zeta) \neq 0$, we have also

$$\frac{p!\tau}{(\|f(\zeta)\|^2 + \tau)^{p+1}} =$$

$$= \frac{1}{(2i\pi)^p} \int_{\tilde{\gamma}_1 + i\mathbf{R}} \cdot \cdot \int_{\tilde{\gamma}_p + i\mathbf{R}} \Gamma(p + 1 - |\underline{s}|) \prod_{j=1}^{p} \Gamma(s_j) \prod_{j=1}^{p} |f_j(\zeta)|^{-2s_j} \tau^{|\underline{s}| - p} ds_1 \cdot \cdot ds_p.$$

where the $\tilde{\gamma}_j$ are real numbers in $]0, 1[$ such that $\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_p < p$ and $|\underline{s}|$ denotes $s_1 + \cdots + s_p$. We may rewrite this as

$$\frac{p!\tau}{(\|f(\zeta)\|^2 + \tau)^{p+1}} =$$

$$= \frac{1}{(2i\pi)^p} \int_{\gamma_1 + i\mathbf{R}} \cdot \cdot \int_{\gamma_p + i\mathbf{R}} \Gamma(1 + |\underline{s}|) \prod_{j=1}^{p} \Gamma(1 - s_j) \prod_{j=1}^{p} |f_j(\zeta)|^{2(s_j - 1)} \tau^{-|\underline{s}|} ds_1 \cdot \cdot ds_p$$

where $\gamma_j := 1 - \tilde{\gamma}_j$. If all $\tilde{\gamma}_j$ are close to zero (that is all γ_j close to 1), it follows as before from Fubini's theorem that, for any $\tau > 0$,

$$\frac{p!(-1)^{p(p-1)/2}\tau}{(2i\pi)^p} \int_V \frac{\overline{\partial f_1} \wedge \cdots \wedge \overline{\partial f_p} \wedge \varphi}{(\|f(\zeta\|^2 + \tau)^{p+1}} =$$

$$= \frac{1}{(2i\pi)^p} \int_{\gamma_1 + i\mathbf{R}} \cdot \cdot \int_{\gamma_p + i\mathbf{R}} \tau^{-|\underline{s}|} \Gamma(|\underline{s}| + 1) \prod_{j=1}^{p} \Gamma(1 - s_j) \frac{J(\underline{s}; \varphi) ds_1 \cdot \cdot ds_p}{s_1 \cdot \cdot s_p} \tag{5.5}$$

where $\underline{\lambda} \mapsto J(\underline{\lambda};\varphi)$ is the function introduced in (1.9). The collection of real hyperplanes in $\mathbf{R}^p$

$$m_{L,0} + \sum_{j=1}^{p} m_{L,j}(x_j + k - 1) = 0\,,\ \ k \in \mathbf{N}\,, x \in \mathbf{R}^p$$

together with the $p+1$ families of hyperplanes $x_1 = k_1\,, k_1 \in \mathbf{N}$, ..., $x_p = k_p\,, k_p \in \mathbf{N}$, $x_1 + \cdots + x_p = -1, -2, \ldots$, determine a decomposition of $\mathbf{R}^p$ into cells. For any $\underline{\gamma}$ interior to each cell, one can define the integral

$$\Xi(\underline{\gamma};\varphi) := \frac{1}{(2i\pi)^p} \int_{\gamma_1 + i\mathbf{R}} \cdot\cdot \int_{\gamma_p + i\mathbf{R}} \tau^{-|\underline{s}|}\Gamma(|\underline{s}|+1) \prod_{j=1}^{p} \Gamma(1-s_j) \frac{J(\underline{s};\varphi) ds_1 \cdot\cdot ds_p}{s_1 \cdot\cdot s_p}\,.$$

Because of the uniform boundedness of $\underline{\lambda} \mapsto J(\underline{\lambda};\varphi)$ on *vertical* strips $K + i\mathbf{R}^p$, where K is any compact in $\mathbf{R}^p$ that lie in one of the cells, the function

$$\underline{\gamma} \mapsto \Xi(\underline{\gamma};\varphi)$$

is constant in each cell of the decomposition (this follows from Cauchy's formula.)

Let us just indicate how to proceed when $p = 2$. In this case, the situation is a little easier since we know that $\lambda_1\lambda_2 J(\underline{\lambda};\varphi)$ is holomorphic near the origin in $\mathbf{C}^2$. Starting with $\underline{\gamma}$ in the interior of the cell $\Delta_0 := [0,1] \times [0,1]$, we proceed as in the proof of Theorem 4.1. We split

$$\Xi(\underline{\gamma};\varphi) = \int_{\gamma_1 + i\mathbf{R}} \int_{\gamma_2 + i\mathbf{R}} \frac{\omega(\underline{s},\tau)}{s_1 s_2}$$

into four terms; two of them correspond to the one dimensional integrals

$$\int_{-\delta_1 + i\mathbf{R}} \mathrm{Res}_{s_2=0} \frac{\omega(\underline{s},\tau)}{s_1 s_2}$$

and

$$\int_{-\delta_2 + i\mathbf{R}} \mathrm{Res}_{s_1=0} \frac{\omega(\underline{s},\tau)}{s_1 s_2}\,.$$

The third one is

$$\int_{-\delta_1 + i\mathbf{R}} \int_{-\delta_2 + i\mathbf{R}} \frac{\omega(\underline{s},\tau)}{s_1 s_2}\,.$$

and corresponds to the value of $\Xi(\gamma;\varphi)$ when $\underline{\gamma}$ lies in the new cell Δ_1 (containing $]-\delta,0[^2$.) Finally, the fourth term is the evaluation of the iterated residue of the meromorphic form $\omega(\underline{s},\tau)/s_1s_2$ with respect to the two divisors $s_1=0$, $s_2=0$. The two first integrals have asymptotic developments in τ which involve local iterated residues for the meromorphic form $\omega(\underline{s},\tau)/s_1s_2$ along pairs of divisors $(\{s_1=0\},D_2)$ at points such that $\Re s_1<0$ (for the first one) and along pairs of divisors $(D_1,\{s_2=0\})$ at points such that $\Re s_2<0$ (for the second one. This follows from Cauchy's formula: we move step by step to the left or the right a vertical line in the complex plane.) It is clear how now one can continue this process, moving from Δ_1 (across a point where x_1+x_2 achieves its minimum in Δ_1) into one of the contiguous cells. The situation is slightly different when one has to cross at a point $(\xi,\eta)\in\mathbf{R}^2$ a line of the form $x_1+x_2=-\rho$, where ρ is a strictly positive rational number (this did not happen in our first step here since the polar set of $\underline{s}\mapsto\omega(\underline{s},\tau)/s_1s_2$ near the origin is just the union of the two axes.) In this case, we use the Jordan lemma to express the corresponding integral as the sum of all iterated residues of the meromorphic form

$$\frac{\omega(\underline{s},\tau)}{s_1s_2}$$

with respect to all pairs of divisors $(\{\lambda_1+\lambda_2=-\rho\},D)$, where D is any hyperplane in the polar set of $\omega(\underline{s},\tau)/s_1s_2$ with slope distinct from -1, at points which lie in one of the half lines in which the line $x_1+x_2=-\rho$ is divided by the point (ξ,η). Note that here, we have a contribution of the form $\tau^\rho\sum_{q=0}^{q-\rho}a_{lq}(\rho)\log^q\tau$, corresponding to an infinite sum of residues.

We therefore have some algorithmic way to get the asymptotic development in terms of the description of the polar set of the meromorphic form

$$\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(s_1+s_2+1)\tau^{-|\underline{s}|}J(\underline{s};\varphi)}{s_1s_2}$$

involved in the integral expression for $\Theta(\tau;\varphi)$. For more details on such a method, one may refer to [20] (where the complete intersection hypothesis is dropped). This completes the proof of our proposition. $\diamondsuit$

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