ANALYTIC RESIDUE THEORY IN THE NON-COMPLETE INTERSECTION CASE [1]

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Abstract

In previous work of the authors and their collaborators (see, *e.g.,* Progress in Math. 114, Birkhäuser (1993)) it was shown how the equivalence of several constructions of residue currents associated to complete intersection families of (germs of) holomorphic functions in $\mathbf{C}^n$ could be profitably used to solve algebraic problems like effective versions of the Nullstellensatz. In this work, the authors explain how such ideas can be transposed to the non-complete intersection situation, leading to an explicit way to construct a Green current attached to a purely dimensional cycle in $\mathbf{P}^n$. This construction extends a previous result of the authors done in the complete intersection case. When the cycle is defined over $\mathbf{Q}$, they give a closed expression for the analytic contribution in the definition of its logarithmic height (as the residue at $\lambda = 0$ of a ζ-function attached to a system of generators of the ideal which defines the cycle). They also introduce an extension of the Cauchy-Weil division process and apply it in order to make explicit the membership of the Jacobian determinant of n elements $f_j \in \mathcal{O}_n$, $j = 1, ..., n$, (which fail to define a regular sequence) in the ideal $(f_1, ..., f_n)$.

0. Introduction.

Let $\mathcal{Z}$ be an effective algebraic cycle of pure dimension $n - d$ in $\mathbf{P}^n(\mathbf{C})$, which corresponds to the homogeneous ideal generated by homogeneous polynomials $P_1, ..., P_m$ in $\mathbf{C}[X_0, X_1, ..., X_n]$. The main result of this paper (Theorem 3.2) is the construction (in terms of the polynomials $P_1, ..., P_m$) of a $(d-1, d-1)$-current valued meromorphic map on $\mathbf{C}$, $\lambda \mapsto \mathbf{G}_\lambda$ such that

$$\mathrm{Res}_{\lambda=0}\,[\mathbf{G}_\lambda]$$

is a current with singular support in $\mathrm{Supp}\,|\mathcal{Z}|$ which satisfies the Green's equation

$$dd^c\mathbf{G} + [\mathcal{Z}] = (\deg \mathcal{Z})(dd^c \log \|\zeta\|^2)^d\,.$$

Such a result extends what we have done in a previous paper [BY2] under the additional assumption that $\mathcal{Z}$ was defined as a complete intersection by the P_j. When the P_j lie in $\mathbf{Z}[X_1, ..., X_n]$, our main Theorem 3.2 leads to the construction (in terms of the polynomials P_j defining the cycle) of an explicit ζ-function whose residue at $\lambda = 0$ is the analytic contribution in the expression of the logarithmic height of the arithmetic cycle $Z(P_1, ..., P_m)$, as defined in [BGS]. We expect such constructions to play a role in the intersection theory developped recently by P. Tworzewski, E. Cygan (see for example [Cyg]).

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In order to realize our objective, it proved to be necessary to extend classical analytic techniques involved in residue calculus from the usual complete intersection (or proper) setting to the improper case. Let us explain here more precisely what are the tools we had to introduce. (In fact, such tools may have their own interest independently of the problem they were introduced for.) They appear as the analytic counterpart to the algebraic approach developped for example in [ScS].

It is a well known fact from multidimensional residue calculus (for example in the spirit of Lipman [Li]) that, given a commutative Noetherian ring $\mathbf{A}$ and a quasi-regular sequence $a_1,\dots,a_n$ of elements in $\mathbf{A}$ such that $\mathbf{A}/(a_1,\dots,a_n)$ is a projective module of finite type, then the all residue symbols

$$\mathrm{Res}\begin{bmatrix} r a_1^{q_1}\cdots a_n^{q_n}\, dr_1\wedge\cdots\wedge dr_n \\ a_1^{q_1+1},\dots,a_n^{q_n+1}\end{bmatrix},\ q\in\mathbf{N}^n,$$

(for $r,r_1,\dots,r_n$ being fixed in $\mathbf{A}$) are independent of q and therefore equal the residue symbol

$$\mathrm{Res}\begin{bmatrix} r\, dr_1\wedge\cdots\wedge dr_n \\ a_1,\dots,a_n\end{bmatrix}.$$

The analytic realization of the residue symbol in the case $\mathbf{A}={}_n\mathcal{O}$, the local ring of germs of holomorphic functions at the origin in $\mathbf{C}^n$, is

$$\mathrm{Res}\begin{bmatrix} h\,dg_1\wedge\cdots\wedge dg_n \\ f_1,\dots,f_n\end{bmatrix} = \lim_{\vec{\epsilon}\to 0}\frac{1}{(2i\pi)^n}\int_{\Gamma_f(\vec{\epsilon})}\frac{h\,dg_1\wedge\cdots\wedge dg_n}{f_1\cdots f_n}, \tag{0.1}$$

where the f_j define a regular sequence in the ring ${}_n\mathcal{O}$ and $\Gamma_f(\vec{\epsilon})$ is the n-dimensional semi-analytic chain $\{|f_1|=\epsilon_1,\dots,|f_n|=\epsilon_n\}$ conveniently oriented (see [GH], chapter 6). In this context, the independence of the symbols

$$\mathrm{Res}\begin{bmatrix} h f_1^{q_1}\cdots f_n^{q_n}\, dg_1\wedge\cdots\wedge dg_n \\ \\ f_1^{q_1+1},\dots,f_n^{q_n+1}\end{bmatrix}$$

with respect to q is, of course, an obvious fact. The advantage dealing with such an analytic realization is that the construction of the objects it involves (namely here residue symbols) may be extended to a less rigid context. We profit from this fact here and, following ideas which were initiated in [BGVY] and [PTY], adopt the current point of view and construct analytic residue symbols attached to a collection $f_1,\dots,f_m$ of germs of holomorphic functions at the origin (which of course may not define a regular sequence) and a pair of algebraic and geometric ponderations. The purpose of the algebraic ponderation is to mimic the construction of residue currents of the form

$$\varphi\to\mathrm{Res}\begin{bmatrix} f_1^{q_1}\cdots f_n^{q_n}\varphi \\ \\ f_1^{q_1+1},\dots,f_n^{q_n+1}\end{bmatrix}, \tag{0.2}$$

φ being a germ of $(n,0)$-smooth test form at the origin; such objects will depend on q if we drop the hypothesis that the sequence $(f_1,\dots,f_n)$ is regular. The key point is the change of section for the representation of the residue symbol in the classical case with the help of the Bochner-Martinelli approach

$$\mathrm{Res}\begin{bmatrix}\varphi\\ f_1,\dots,f_n\end{bmatrix}=\lim_{\epsilon\to 0}\frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi\epsilon)^n}\int_{\|f\|_\rho^2=\epsilon}\Big(\sum_{k=1}^{n}(-1)^{k-1}\bigwedge_{\substack{l=1\\ l\neq k}}^{n}\overline{\partial}(\rho_l^2\overline{f_l})\Big)\wedge\varphi\,, \quad (0.3)$$

where $\rho_1^2,\dots,\rho_n^2$ are germs of smooth strictly positive functions and

$$\|f\|_\rho^2:=\rho_1^2|f_1|^2+\cdots+\rho_n^2|f_n|^2\,.$$

When $f_1,\dots,f_n$ do not define a regular sequence anymore, one may still define the action of a $(0,n)$ germ of current thanks to the Bochner-Martinelli construction (0.3), but the constructions will of course depend of the geometric ponderation ρ.

We will construct such residual objects in section 1 of this paper. Though the currents we introduce will in general not be closed, they will appear as "quotients" in the division of some positive closed currents (dependent on the ponderations) by the df_j, this is essentially the same as in the complete intersection case, where we have the well known factorisation formula for the integration current $\delta_{[V(f)]}$ (with multiplicities) attached to the cycle corresponding to the f_j:

$$\delta_{[V(f)]}(\varphi)=\mathrm{Res}\begin{bmatrix}\varphi\wedge df_1\wedge\cdots\wedge df_p\\ f_1,\dots,f_p\end{bmatrix}$$

(here $f_1,\dots,f_p$ define a germ of complete intersection and the action of the residue symbol corresponds to the action of the Coleff-Herrera current).

What seems to us as an interesting point (besides the fact that such currents are involved in the proof of our main Theorem 3.2) is that they also play a significant role in the realization of division-interpolation formulas in the spirit of Cauchy-Weil's formula. The fact that in the classical case, the Cauchy-Weil formula can be understood within the general frame of an algebraic theory for residue calculus (see for example [BoH], [BY3]) gives us some hope that the generalizations we propose here (see Theorem 2.1) could be also interpreted from an algebraic point of view.

As an illustration of the range of application of such techniques, we also study in section 2 a division problem inspired by a result (in the homogeneous algebraic case) stated by E. Netto [Net], and proved later in a constructive way in [Sp]: if $P_1,\dots,P_n$ are n homogeneous polynomials which simultaneously vanish at some point in $\mathbf{C}^n\setminus\{0\}$, then, there is an explicit division procedure (based on the use of the Euler identity) in order to express the Jacobian determinant of $(P_1,...,P_n)$ in the ideal generated by the P_j. It was kindly pointed to us by W. Vasconscelos that when $P_1,...,P_n$ are n arbitrary polynomials in n variables, then the Jacobian determinant J of $(P_1,...,P_n)$ transports the top-radical of the ideal $I=I(P_1,...,P_n)$ into I itself, which implies indeed that J lies in $I(P_1,...,P_n)$ if and only if the system of equations $\{P_1 = ... = P_n = 0\}$ has no isolated zeros ([Vas1], [Vas2]). Inspired by a first draft of this manuscript and the algebraic approach from [ScS] and [Vas1], M. Hickel proved recently that the local version of this result holds: the Jacobian determinant of n germs $f_1, ..., f_n$ in $\mathcal{O}_n$ lies in $(f_1, ..., f_n)$ if and only if the sequence $(f_1, ..., f_n)$ fails to be regular in $\mathcal{O}_n$ ([H]). We present in Section 2 of this paper a division process in order to solve such a membership problem, that is, write explicitely the Jacobian determinant of $f_1, ..., f_n$ in $I(f_1, ..., f_n)$, when $\sqrt{I(f_1, \dots, f_n)} = \sqrt{I(f_1, \dots, f_d)}$ for some $d < n$ or when the analytic spread of $(f_1, \dots, f_n)$ is strictly less than n (see Proposition 2.1 and Theorem 2.2).

We dedicate this work to the memory of Gian-Carlo Rota, whose review [Ro] of our book [BGVY] gave us encouragement to continue our research in this subject.

1. Residue currents in the non-complete intersection case.

Let $m \geq 1$ be a positive integer, U an open subset in $\mathbf{C}^n$, and $s = (s_1, \dots, s_m)$ a vector of m C^1 complex-valued functions in U. For any ordered subset $\mathcal{I} = \{i_1, \dots, i_r\} \subset \{1, \dots, m\}$ with cardinal $r \leq \min(m, n)$, we will denote by $\Omega(s; \mathcal{I})$ the differential form

$$\Omega(s;\mathcal{I}) = \sum_{k=1}^{r} (-1)^{k-1} s_{i_k} \bigwedge_{\substack{l=1 \\ l \neq k}}^{r} ds_{i_l} \, .$$

Let now $f_1, \dots, f_m$ be m complex-valued holomorphic functions of n variables in the open set U, such that the analytic variety $V(f) := \{f_1 = \dots = f_m = 0\}$ has codimension d (we do not assume here that $V(f)$ is purely dimensional). Let $q_1, \dots, q_m$ be m positive integers and $\rho_1, \dots, \rho_m$ m non vanishing real analytic functions in V, and $\epsilon > 0$, then, as an example of vector $s = (s_1, \dots, s_m)$, we consider

$$s^{q,\rho,\epsilon} = \frac{1}{\epsilon} (\rho_1^2 \overline{f_1} |f_1|^{2q_1}, \dots, \rho_m^2 \overline{f_m} |f_m|^{2q_m}) \, .$$

We also define

$$\|f\|_{q,\rho}^2 = < s^{q,\rho,1}, f > = \sum_{k=1}^{m} \rho_k^2 |f_k|^{2(q_k+1)} \, .$$

We have the following lemma

Lemma 1.1. *For any ordered subset $\mathcal{I} \subset \{1, \dots, m\}$ with cardinal $r \leq \min(m, n)$, for any $(n, n-r)$ test form φ with coefficients in $\mathcal{D}(U)$, the limit*

$$\mathrm{Res} \begin{bmatrix} \varphi \\ f_{i_1}, \dots, f_{i_r} \\ f_1, \dots, f_m \end{bmatrix}^{q,\rho} = \lim_{\epsilon \to 0} \frac{(-1)^{\frac{r(r-1)}{2}} (r-1)!}{(2i\pi)^r} \int_{\|f\|_{\rho,q}^2 = \epsilon} \Omega(s^{q,\rho,\epsilon}; \mathcal{I}) \wedge \varphi \qquad (1.1)$$

exists and

$$\varphi \mapsto \mathrm{Res} \begin{bmatrix} \varphi \\ f_{i_1}, \dots, f_{i_r} \\ f_1, \dots, f_m \end{bmatrix}^{q,\rho}$$

defines a $(0,r)$ current in U. This current is 0 when $r < \operatorname{codim} V(f)$ and, for any $(n,n-r)$ test form φ and any holomorphic function h in U, we have that

$$h=0 \text{ on } V(f) \implies \operatorname{Res}\begin{bmatrix} \overline{h}\varphi \\ f_{i_1},\dots,f_{i_r} \\ f_1,\dots,f_m \end{bmatrix}^{q,\rho} = 0$$

$$\Big(\prod_{l=1}^{r} f_{i_l}^{q_{i_l}}\Big) h_z \in \overline{(f_1^{q_1+1},\dots,f_m^{q_m+1})^r \mathcal{O}_z} \quad \forall z \in V(f) \implies \operatorname{Res}\begin{bmatrix} h\varphi \\ f_{i_1},\dots,f_{i_r} \\ f_1,\dots,f_m \end{bmatrix}^{q,\rho} = 0\,, \tag{1.2}$$

where we denoted by $\overline{I}$ the integral closure of an ideal I and by $(f_1^{q_1+1},\dots,f_m^{q_m+1})^r\mathcal{O}_z$ the r-th power of the ideal in $\mathcal{O}_z$ which is generated by the germs at z of the $f_j^{q_j+1}$.

Proof. The proof of this result was given in [PTY] when $q=0$ and $\rho_j \equiv 1$ for any j. Since the contributions of the weights q and ρ do not substantially affect the proof, we will just sketch it here. The idea is to compute, when φ is fixed, the Mellin transform of the function

$$\epsilon \mapsto I^{q,\rho}(\varphi;\mathcal{I};\epsilon) = \frac{(-1)^{\frac{r(r-1)}{2}}(r-1)!}{(2i\pi)^r} \int_{\|f\|^2_{\rho,q}=\epsilon} \Omega(s^{q,\rho,\epsilon};\mathcal{I}) \wedge \varphi,$$

that is, the function

$$\lambda \mapsto J^{q,\rho}(\varphi;\mathcal{I};\lambda) = \lambda \int_0^\infty I(\varphi;\epsilon)\epsilon^{\lambda-1} d\epsilon$$

defined (and holomorphic) in the half-plane $\operatorname{Re}\lambda > r+1$. One has

$$J^{q,\rho}(\varphi;\mathcal{I};\lambda) = \frac{(-1)^{\frac{r(r-1)}{2}}(r-1)!\lambda}{(2i\pi)^r} \int \|f\|^{2(\lambda-r)}_{q,\rho} \overline{\partial} \log \|f\|^2_{q,\rho} \wedge \Omega(s^{q,\rho,1};\mathcal{I}) \wedge \varphi\,. \tag{1.3}$$

Since the result stated in the lemma is local, we can prove it when the support of φ is contained in some arbitrary small neighborhhood of a point $z_0 \in V(f)$ (near any other point, the limit (1.1) equals 0, as a consequence, for example, of the coarea formula in [Fe]). As in our previous work ([BGVY, BY, PTY]), we construct an analytic n dimensional manifold $\mathcal{X}_{z_0}$, a neighborhhood $W(z_0)$ of z_0, a proper map $\pi : \mathcal{X}_{z_0} \leftarrow W(z_0)$ which realizes a local isomorphism between $W(z_0) \setminus \{f_1 \cdots f_m = 0\}$ and $\mathcal{X}_{z_0} \setminus \pi^{-1}(\{f_1\cdots f_m = 0\})$, such that in local coordinates on $\mathcal{X}_{z_0}$ (centered at a point x), one has, in the corresponding local chart U_x around x,

$$f_j \circ \pi(t) = u_j(t) t_1^{\alpha_{j1}} \cdots t_n^{\alpha_{jn}} = u_j(t) t^{\alpha_j},\ j=1,\dots,m,$$

where the u_j are non vanishing holomorphic functions and at least one of the monomials $t^{(q_j+1)\alpha_j} = \mu(t)$ divides any $t^{(q_k+1)\alpha_k}$, $k=1,\dots,m$. Note that the normalized blow-up of the ideal $(f_1^{q_1+1},\dots,f_m^{q_m+1})\mathcal{O}_{z_0}$, as used in [Te], is not enough for us, since we need to put ourselves in the normal crossing case in order to prove the existence of the limit (1.1). Note also that any coordinate t_k which divides μ divides all the $\pi^* f_j$, $j=1,...,m$. Let us define the formal expression

$$\Theta_\lambda = \lambda \|f\|^{2(\lambda-r)}_{q,\rho} \overline{\partial} \log \|f\|^2_{q,\rho} \wedge \Omega(s^{q,\rho,1};\mathcal{I})\,,$$

λ being a complex parameter. If we express this differential form in local coordinates t and profit from the fact that μ divides all $(\pi^* f_j)^{q_j+1}$, $j=1,...,m$, we get

$$\pi^*\Theta_\lambda = \lambda \frac{|a\mu|^{2\lambda}}{\mu^r}\Big(\prod_{l=1}^{r}(\pi^* f_{i_l})^{q_{i_l}}\Big)\Big(\vartheta + \varpi \wedge \frac{\overline{\partial \mu}}{\overline{\mu}}\Big)\,, \tag{1.4}$$

where ϑ and ϖ are smooth forms of respective type $(0,r)$ and $(0,r-1)$ and a is a non vanishing function. Since $J^{q,\rho}(\varphi;\mathcal{I};\lambda)$ is a combination of terms of the form

$$\int_{U_x} \pi^*\Theta_\lambda \wedge \psi\pi^*\varphi, \tag{1.5}$$

where $x\in\mathcal{X}_{z_0}$, ψ is an element of a partition of unity for $\pi^*(\operatorname{Supp}\varphi)$ and $\frac{\overline{\partial\mu}}{\overline{\mu}}$ is a linear combination of the $\frac{d\overline{t_l}}{\overline{t_l}}$, $l=1,\dots,n$. We conclude from the techniques based on integration by parts developped for example in [BGVY], chapter 3, section 2, that

$$\lambda \mapsto J^{q,\rho}(\varphi;\mathcal{I};\lambda)$$

can be continued as a meromorphic function in $\mathbf{C}$, whose poles are strictly negative rational numbers. When h is a holomorphic function in U which vanishes on $V(f)$, all coordinates t that divide μ divide also π^*h since they divide all π^*f_j, $j=1,\dots,m$. It follows that, for any test form φ, $J^{q,\rho}(\overline{h}\varphi;\mathcal{I};0)=0$, since the singularities of the differential form $\pi^*\Theta_\lambda\wedge\psi\pi^*(\overline{h}\varphi)$ have no antiholomorphic factor. Let us suppose now that the germ of h at z_0 is such that

$$\big(\prod_{l=1}^{r} f_{i_l}^{q_{i_l}}\big)h_{z_0} \in \overline{(f_1^{q_1+1},\dots,f_m^{q_m+1})^r\mathcal{O}_{z_0}}\,.$$

It follows from the valuative criterion [LeT] that μ^r divides

$$\Pi_h = \Big(\prod_{l=1}^{r}(\pi^* f_{i_l})^{q_{i_l}}\Big)\pi^* h\,.$$

Thus, the singularities of the differential form $\pi^*\Theta_\lambda\wedge\psi\pi^*(h\varphi)$ have no holomorphic factor. Hence, in this case, we can again conclude that $J^{q,\rho}(h\varphi;\mathcal{I};0)=0$.

On the other hand, we know from ([Bjo1], 6.1.19) that for any $z_0\in V(f)$, there is a strictly positive integer N_{z_0} and differential operators $\mathcal{Q}_{z_0,j}(\zeta,\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\overline{\zeta}})$ with coefficients in $\mathcal{O}_{z_0}$ such that

$$\Big[\lambda^{N_{z_0}} - \sum_{j=1}^{N_{z_0}} \lambda^{N_{z_0}-j}\mathcal{Q}_{z_0,j}(\zeta,\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\overline{\zeta}})\Big]\ \|f\|_{q,\rho}^{2\lambda} = 0\,,$$

where this is an identity between two distribution-valued meromorphic functions of λ in a neighborhood of z_0. With the help of this identity we can prove, as in [BaM,Bjo2], that the meromorphic continuation of the function $\lambda\mapsto J^{q,\rho}(\varphi;\mathcal{I};\lambda)$ has rapid decrease on vertical lines in the complex plane when λ tends to ∞. Therefore, we can invert the Mellin transform and obtain the existence of the limit when $\epsilon \to 0$ of the function $\epsilon \mapsto I^{q,\rho}(\varphi;\mathcal{I};\epsilon)$. We also have $I^{q,\rho}(\varphi;\mathcal{I};0) = J^{q,\rho}(\varphi;\mathcal{I};0)$. In order to prove that the currents we just constructed are zero if $r < d$ we proceed as follows. Assume that $r < d$ and choose a test form $\varphi \in \mathcal{D}^{n,n-r}(W(z_0))$. One can rewrite φ as

$$\varphi = \sum_{1\le j_1<\cdots<j_{n-r}\le n} \varphi_{j_1,\ldots,j_{n-r}} d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge \bigwedge_{l=1}^{n-r} \overline{d\zeta_{j_l}} \,.$$

For dimensionality reasons, each differential form $\bigwedge_{l=1}^{n-r} \overline{d\zeta_{j_l}}$ is zero when restricted to the $n-d$-dimensional analytic variety $V(f)$. This implies that, given a local chart U_x around some point x on the analytic manifold $\mathcal{X}$, the differential form $\pi^* \bigwedge_{l=1}^{n-r} \overline{d\zeta_{j_l}}$ (which has antiholomorphic functions as coefficients) vanishes on the analytic variety $\{\mu(t) = 0\}$, where μ is the distinguished monomial corresponding to the local chart. Every conjugate coordinate $\bar{t}_k$ such that t_k divides μ, divides each coefficient of $\pi^* \bigwedge_{l=1}^{n-r} \overline{d\zeta_{j_l}}$ which does not contain $d\bar{t}_k$. This implies that for any local chart U_x, the differential form $\pi^*\Theta_\lambda \wedge \psi\pi^*(\varphi)$ appearing in the integral (1.5) related to this chart contains only holomorphic singularities (such singularities arise from logarithmic derivatives and therefore are cancelled by the corresponding terms coming from $\pi^*\varphi$). This completes the proof. ◇

We can combine these currents with the differential forms df_j, in order to construct certain closed positive currents $[f]_r^{q,\rho}$, $r = d, \ldots, \min(m,n)$. Among them, the currents that corresponds to $r = d$ are related (as we shall see later) to the integration current (with multiplicities) on the analytic cycle defined by the f_j. The other ones will usually be supported on the embedded components of the cycle, provided q is chosen conveniently.

Lemma 1.2. *Let $U, f_1, \ldots, f_m, q, \rho$ be as in Lemma 1.1, and $d \le r \le \min(m,n)$, then the (r,r) current*

$$\varphi \mapsto \sum_{1\le i_1<i_2<\ldots<i_r\le m} \Big(\prod_{l=1}^{r}(q_{i_l}+1)\Big) \operatorname{Res} \begin{bmatrix} df_{i_1} \wedge \cdots \wedge df_{i_r} \wedge \varphi \\ f_{i_1}, \ldots, f_{i_r} \\ f_1, \ldots, f_m \end{bmatrix}^{q,\rho} \tag{1.6}$$

is a closed positive current $[f]_r^{q,\rho}$ supported by $V(f)$. The action of this current on a $(n-r, n-r)$ test form can be also expressed as the residue at $\lambda = 0$ of the meromorphic function of λ

$$\frac{(r-1)!}{(2\pi i)^r} \int_U \|f\|_{q,\rho}^{2(\lambda-r-1)} \bar\partial \|f\|_{q,\rho}^2 \wedge \partial \|f\|_{q,\rho}^2 \wedge \Big[\sum_{\substack{j_1<\ldots<j_{r-1}\\ 1\le j_l\le m}} \bigwedge_{l=1}^{r-1} \bar\partial(\rho_{j_l} \overline{f_{j_l}}^{q_{j_l}+1}) \wedge \partial(\rho_{j_l} f_{j_l}^{q_{j_l}+1}) \Big] \wedge \varphi \tag{1.7}$$

Proof. First we give the proof of this lemma when the functions ρ_j are constant. We have in this case

$$\partial \|f\|_{q,\rho}^2 = \sum_{j=1}^{m} (q_j+1)\rho_j^2 |f_j|^{2q_j} \, df_j$$

and

$$\overline{\partial} s_j^{q,\rho,1} = \sum_{j=1}^{m} (q_j+1)\rho_j^2 |f_j|^{2q_j}\, \overline{df_j}\,,\ j=1,\dots,m\,.$$

An immediate algebraic computation shows that, for any $(n-r,n-r)$ test form φ,

$$\begin{aligned}
&\Big[\sum_{\substack{i_1<\dots<i_r\\ 1\le i_l\le m}} \Big(\prod_{l=1}^{r}(q_{i_l}+1)\Big)\Omega(s^{q,\rho,1};\{i_1,\dots,i_r\}) \wedge \bigwedge_{l=1}^{r} df_{i_l}\Big] \wedge \varphi = \\
&= (-1)^r \partial \|f\|^2_{q,\rho} \wedge \Big[\sum_{\substack{j_1<\dots<j_{r-1}\\ 1\le j_l\le m}} \bigwedge_{l=1}^{r-1} \overline{\partial}(\rho_{j_l} \overline{f_{j_l}}^{q_{j_l}+1}) \wedge \bigwedge_{l=1}^{r-1} \partial(\rho_{j_l} f_{j_l}^{q_{j_l}+1})\Big] \wedge \varphi\,.
\end{aligned} \tag{1.8}$$

Let now, for $\epsilon>0$,

$$\Phi(\epsilon) = \frac{\gamma_r}{\epsilon^r} \int_{\|f\|^2_{q,\rho}=\epsilon} \Big[\sum_{\substack{i_1<\dots<i_r\\ 1\le i_l\le m}} \Big(\prod_{l=1}^{r}(q_{i_l}+1)\Big)\Omega(s^{q,\rho,1};\{i_1,\dots,i_r\}) \wedge \bigwedge_{l=1}^{r} df_{i_l}\Big] \wedge \varphi\,,$$

where

$$\gamma_r := \frac{(-1)^{\frac{r(r-1)}{2}}(r-1)!}{(2i\pi)^r}\,.$$

We know from Lemma 1.1 that the limit of $\Phi(\epsilon)$ when $\epsilon\to 0$ exists and equals (by definition of the residue symbols) exactly $[f]_r^{q,\rho}$. This implies that the function defined on $]0,\infty[$ by

$$\tau \mapsto \Psi(\tau) = \tau\gamma_r r \int_0^\infty \frac{\epsilon^{r-1}\Phi(\epsilon)d\epsilon}{(\epsilon+\tau)^{r+1}}$$

also has a limit at 0, which equals $\Psi(0)=\Phi(0)=[f]_r^{q,\rho}(\varphi)$. Using the Fubini and Lebesgue theorems, one can show that for any $\tau>0$,

$$\begin{aligned}
\Psi(\tau) &= \tau r \gamma_r \int_U \frac{\overline{\partial}\|f\|^2_{q,\rho} \wedge \Big[\sum\limits_{\substack{i_1<\dots<i_r\\ 1\le i_l\le m}} \Big(\prod\limits_{l=1}^{r}(q_{i_l}+1)\Big)\Omega(s^{q,\rho,1};\{i_1,\dots,i_r\}) \wedge \bigwedge\limits_{l=1}^{r} df_{i_l}\Big] \wedge \varphi}{\|f\|^2_{q,\rho}(\|f\|^2_{q,\rho}+\tau)^{r+1}} \\
&= \frac{\tau r!}{(2\pi i)^r} \int_U \frac{\overline{\partial}\|f\|^2_{q,\rho} \wedge \partial\|f\|^2_{q,\rho} \wedge \Big[\sum\limits_{\substack{j_1<\dots<j_{r-1}\\ 1\le j_l\le m}} \bigwedge\limits_{l=1}^{r-1} \overline{\partial}(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1}) \wedge \partial(\rho_{j_l} f_{j_l}^{q_{j_l}+1})\Big] \wedge \varphi}{\|f\|^2_{q,\rho}(\|f\|^2_{q,\rho}+\tau)^{r+1}}
\end{aligned} \tag{1.9}$$

(note that the integrals in the right-hand side of (1.9) are absolutely convergent, which justifies our use of those theorems to perform the computation of $\Psi(\tau)$). Since $\Psi(\tau)$ corresponds to the action on φ of a positive current (just look at the second equality in

(1.9)), the current $\varphi \mapsto [f]_r^{q,\rho}(\varphi) = \Phi(0) = \Psi(0)$ is positive. On the other hand, we have also

$$\begin{aligned}\Phi(\epsilon) &= \frac{(r-1)!}{(2\pi i\epsilon)^r}\int_{\|f\|^2_{q,\rho}=\epsilon} \partial\|f\|^2_{q,\rho}\wedge\Big[\sum_{\substack{j_1<\dots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}\overline{\partial}(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge\partial(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\varphi\\ &= -\frac{(r-1)!}{(2\pi i\epsilon)^r}\int_{\|f\|^2_{q,\rho}=\epsilon} \overline{\partial}\|f\|^2_{q,\rho}\wedge\Big[\sum_{\substack{j_1<\dots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}\overline{\partial}(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge\partial(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\varphi\,.\end{aligned} \tag{1.10}$$

Since the ρ_j are here supposed constant, the differential form

$$\sum_{\substack{j_1<\dots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}\overline{\partial}(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge\partial(\rho_{j_l}f_{j_l}^{q_{j_l}+1})$$

is d-closed. It follows from Stokes's theorem that

$$\begin{aligned}&\int_{\|f\|^2_{q,\rho}=\epsilon} \partial\|f\|^2_{q,\rho}\wedge\Big[\sum_{\substack{j_1<\dots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}\overline{\partial}(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge\partial(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\partial\psi=\\ &=\int_{\|f\|^2_{q,\rho}=\epsilon} \overline{\partial}\|f\|^2_{q,\rho}\wedge\Big[\sum_{\substack{j_1<\dots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}\overline{\partial}(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge\partial(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\overline{\partial}\xi=0\end{aligned}$$

for any $(n-r-1,n-r)$ (resp. $(n-r,n-r-1)$) test form ψ (resp. ξ). Therefore, we have, if $\varphi=\partial\psi$ or $\varphi=\overline{\partial}\xi$, $\Phi(0)=\lim_{\epsilon\to 0}\Phi(\epsilon)=[f]_r^{q,\rho}(\varphi)=0$, which shows that the current $[f]_r^{q,\rho}$ is closed. Thus, we have proved that if the ρ_j are constants, the current $[f]_r^{q,\rho}$ is closed and positive.

We now come back to the general case. The Mellin transform of the function

$$\Phi(\epsilon)=\frac{\gamma_r}{\epsilon^r}\int_{\|f\|^2_{q,\rho}=\epsilon}\Big[\sum_{\substack{i_1<\dots<i_r\\ 1\le i_l\le m}}\Big(\prod_{l=1}^{r}(q_{i_l}+1)\Big)\Omega(s^{q,\rho,1};\{i_1,\dots,i_r\})\wedge\bigwedge_{l=1}^{r}df_{i_l}\Big]\wedge\varphi$$

is

$$\begin{aligned}&\lambda\int_0^\infty \epsilon^{\lambda-1}\Phi(\epsilon)d\epsilon=\\ &=\lambda\gamma_r\int_U ||f||^{2(\lambda-r-1)}\overline{\partial}||f||^2_{q,\rho}\wedge\Big[\sum_{\substack{i_1<\dots<i_r\\ 1\le i_l\le m}}\Big(\prod_{l=1}^{r}(q_{i_l}+1)\Big)\Omega(s^{q,\rho,1};\mathcal{I})\wedge\bigwedge_{l=1}^{r}df_{i_l}\Big]\wedge\varphi\end{aligned} \tag{1.11}$$

If we express this function using the same resolution of singularities that we used in the proof of Lemma 1.1 and use the algebraic relation (1.8), we see that the value at $\lambda = 0$ of this function is the same than the value at $\lambda = 0$ of the function of λ

$$\frac{\lambda(r-1)!}{(2i\pi)^r}\int_U \|f\|^{2(\lambda-r-1)}\overline{\partial}\|f\|^2_{q,\rho}\wedge\partial\|f\|^2_{q,\rho}\wedge\Big[\sum_{\substack{j_1<\cdots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}\overline{\partial}(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge\partial(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\varphi$$

(any term where the differentiation of one of the ρ_j is involved does not contribute to the value at $\lambda = 0$, since, when we express it in local coordinates on the local chart after resolution of singularities, the integrand contains only holomorphic factors in its denominator). This function is the Mellin transform of the following function of $\epsilon > 0$,

$$\epsilon\mapsto\frac{(r-1)!}{(2\pi i\epsilon)^r}\int_{||f||^2_{q,\rho}=\epsilon}\partial||f||^2_{q,\rho}\wedge\Big[\sum_{\substack{j_1<\cdots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}\overline{\partial}(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge\partial(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\varphi\,.$$

Using the same argument preceeding (1.9), one sees that the value of $\widetilde{\Phi}$ at $\epsilon = 0$, which is well-defined, equals the value at $\tau = 0$ of the function

$$\widetilde{\Psi}(\tau)=\frac{\tau r!}{(2\pi i)^r}\int_U\frac{\overline{\partial}\|f\|^2_{q,\rho}\wedge\partial\|f\|^2_{q,\rho}\wedge\Big[\sum\limits_{\substack{j_1<\cdots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge\limits_{l=1}^{r-1}(\overline{\partial}(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge\partial(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\varphi}{\|f\|^2_{q,\rho}(\|f\|^2_{q,\rho}+\tau)^{r+1}}\,.$$

Since $\widetilde{\Phi}(0)=\widetilde{\Psi}(0)=[f]_r^{q,\rho}(\varphi)$, the last current is positive as a limit of positive smooth currents, as seen earlier in (1.9). As above, note that the value at $\lambda = 0$ of the function defined by (1.11) is the same as the value at $\lambda = 0$ of the function

$$\frac{\lambda(r-1)!}{(2i\pi)^r}\int_U \|f\|^{2(\lambda-r-1)}\overline{\partial}\|f\|^2_{q,\rho}\wedge\partial\|f\|^2_{q,\rho}\wedge\Big[\sum_{\substack{j_1<\cdots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}d(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge d(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\varphi$$

This function is the Mellin transform of the function defined for $\epsilon > 0$ by

$$\begin{aligned}\widetilde{\Phi}(\epsilon)&=\frac{(r-1)!}{(2\pi i\epsilon)^r}\int_{||f||^2_{q,\rho}=\epsilon}\partial||f||^2_{q,\rho}\wedge\Big[\sum_{\substack{j_1<\cdots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}d(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge d(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\varphi\\ &=-\frac{(r-1)!}{(2\pi i\epsilon)^r}\int_{||f||^2_{q,\rho}=\epsilon}\overline{\partial}||f||^2_{q,\rho}\wedge\Big[\sum_{\substack{j_1<\cdots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}d(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge d(\rho_{j_l}f_{j_l}^{q_{j_l}+1})\Big]\wedge\varphi\,.\end{aligned}$$

Since the differential form

$$\sum_{\substack{j_1<\cdots<j_{r-1}\\ 1\le j_l\le m}}\bigwedge_{l=1}^{r-1}d(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1})\wedge d(\rho_{j_l}f_{j_l}^{q_{j_l}+1})$$

is closed, it follows from Stokes's theorem that

$$\int_{\|f\|^2_{q,\rho}=\epsilon} \partial\|f\|^2_{q,\rho} \wedge \Big[\sum_{\substack{j_1<\dots<j_{r-1}\\ 1\leq j_l\leq m}} \bigwedge_{l=1}^{r-1} (d(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1}) \wedge d(\rho_{j_l} f_{j_l}^{q_{j_l}+1})\Big] \wedge \partial\psi =$$

$$= \int_{\|f\|^2_{q,\rho}=\epsilon} \overline{\partial}\|f\|^2_{q,\rho} \wedge \Big[\sum_{\substack{j_1<\dots<j_{r-1}\\ 1\leq j_l\leq m}} \bigwedge_{l=1}^{r-1} (d(\rho_{j_l}\overline{f_{j_l}}^{q_{j_l}+1}) \wedge d(\rho_{j_l} f_{j_l}^{q_{j_l}+1})\Big] \wedge \overline{\partial}\xi = 0$$

for any $(n-r-1, n-r)$ (resp. $(n-r, n-r-1)$) test form ψ (resp. ξ). Therefore, the current $\varphi \mapsto [f]^{q,\rho}_r(\varphi) = \widetilde{\Phi}(0) = \lim_{\epsilon\to 0} \widetilde{\Phi}(\epsilon)$ is closed. This completes the proof. $\diamondsuit$

2. Interpolation-Division formulas.

Let $m \in \mathbf{N}^*$, U an open set in $\mathbf{C}^n$, and $f_1, \dots, f_m$, m holomorphic complex-valued functions in U. Let $s_1, \dots, s_m$ be m C^1 complex-valued functions in U. Let $< s, f >$ be the function defined in U as

$$< s(\zeta), f(\zeta) >=< s, f > (\zeta) := \sum_{j=1}^{m} s_j(\zeta) f_j(\zeta)\,.$$

Let $u_1, \dots, u_m$ be m C^1 $(1,0)$forms in U. Consider the formal differential form in U defined as

$$\Xi(\lambda;\zeta,u) =< s, f >^{\lambda-1} \sum_{j=1}^{m} s_j du_j\,.$$

One has, if ψ_1 is any $(n-1,0)$ form in ζ,

$$d_\zeta\Xi(\lambda;\zeta,u)\wedge\psi_1 =< s, f >^{\lambda-1} \left((\lambda-1)\left[\frac{d < s, f > \wedge \sum_{j=1}^{m} s_j du_j}{< s, f >^2}\right] + \sum_{j=1}^{m} ds_j \wedge du_j \right) \wedge \psi_1\,.$$

Therefore, if ψ_r is any $(n-r,0)$ differential form in ζ,

$$\frac{(-1)^{\frac{r(r-1)}{2}}}{r!}(d_\zeta\Xi(\lambda;\zeta,u))^r \wedge \psi_r =$$

$$=< s, f >^{r(\lambda-1)} \sum_{\substack{i_1<\dots<i_r\\ 1\leq i_l\leq m}} \Big[\bigwedge_{l=1}^{r} ds_{i_l} + (\lambda-1)\frac{d < s, f >}{< s, f >} \wedge \Omega(s;\mathcal{I})\Big] \wedge \big(\bigwedge_{l=1}^{r} du_{i_l}\big) \wedge \psi_r \tag{2.1}$$

where, for any ordered subset $\mathcal{I} = \{i_1, \dots, i_r\}$ of $\{1, ..., m\}$, $\Omega(s;\mathcal{I})$ has been defined in Section 1. The term containing λ as a factor in the development of $(d_\zeta\Xi(\lambda;\zeta,u))^r \wedge \psi_r$ is

$$(-1)^{\frac{r(r-1)}{2}} r!\ \lambda < s, f >^{r(\lambda-1)} \frac{d < s, f >}{< s, f >} \wedge \sum_{\substack{i_1<\dots<i_r\\ 1\leq i_l\leq m}} \Omega(s;\mathcal{I}) \wedge \big(\bigwedge_{l=1}^{r} du_{i_l}\big) \wedge \psi_r\,. \tag{2.2}$$

In particular, when $s = s^{q,\rho,1}$ as in Section 1, this coefficient is exactly

$$(-1)^{\frac{r(r-1)}{2}} r!\lambda \|f\|_{q,\rho}^{2r(\lambda-1)} \sum_{\substack{i_1<\ldots<i_r \\ 1\le i_l\le m}} \frac{\overline{\partial}\|f\|_{q,\rho}^2}{\|f\|_{q,\rho}^2} \wedge \Omega(s^{q,\rho,1};\{i_1,\ldots,i_r\}) \wedge \Big(\bigwedge_{l=1}^{r} du_{i_l}\Big) \wedge \psi_r\,. \tag{2.3}$$

The following result is a variant of a division formula that appears in [BGVY, DGSY].

Theorem 2.1. *Let $f_1,\ldots,f_m$ be m holomorphic functions in some neighborhood U of the origin in $\mathbf{C}^n$, $n > m$. Let $q \in \mathbf{N}^m$ and $\rho_1,\ldots,\rho_m$ m real-analytic functions non vanishing in U. Suppose that $[g_{jk}]_{\substack{1\le j\le m\\ 1\le k\le n}}$ is a matrix of holomorphic functions in $U\times U$ such that*

$$f_j(z) - f_j(\zeta) = \sum_{k=1}^{n} g_{jk}(z,\zeta)(z_k-\zeta_k)\,,\ j=1,\ldots,m,$$

and let

$$G_j(z,\zeta) = \sum_{k=1}^{n} g_{jk}(z,\zeta)d\zeta_k\,,\ j=1,\ldots,m\,.$$

Let φ be a test function with compact support in U which is identically equal to 1 in some neighborhood $\widetilde{U}$ of the origin, and σ a C^1 n-valued function of $2n$ variables (z,ζ), defined in $\widetilde{U}\times W$, where W is a neighborhood of $\operatorname{supp}(d\varphi)$, holomorphic in z, and such that, for any $z\in\widetilde{U}$,

$$d\varphi(\zeta)\neq 0 \Longrightarrow \sum_{k=1}^{n}\sigma_k(z,\zeta)(\zeta_k-z_k)=1.$$

For any function h holomorphic in U, let the function $T_0^{q,\rho}h$ be defined in $\widetilde{U}$ by

$$\begin{gathered} T_0^{q,\rho}h(z) = -\sum_{d\le r\le m}\ \sum_{\substack{i_1<\ldots<i_{n-r}\\ 1\le i_l\le n}}\ \sum_{\substack{j_1<\ldots<j_r\\ 1\le j_s\le m}} \\ \left(\gamma_{n-r}\,\mathrm{Res}\begin{bmatrix} h d\varphi\wedge\Omega(\sigma(z,\zeta);\mathcal{I})\wedge\Big(\bigwedge_{l=1}^{n-r} d\zeta_{i_l}\Big)\wedge\bigwedge_{s=1}^{r} G_{j_s}(z,\zeta) \\ f_{j_1},\ldots,f_{j_r} \\ f_1,\ldots,f_m\end{bmatrix}^{q,\rho}\right) \end{gathered} \tag{2.4}$$

where, $\gamma_t = \frac{(-1)^{\frac{t(t-1)}{2}}(t-1)!}{(2\pi i)^t}$, $t\in\mathbf{N}$, and the action of the residual currents is computed with respect to the ζ-variables. Then, $T_0^{q,\rho}h$ has the property that the germ $(h-T_0^{q,\rho}h)_0 \in (f_1,\ldots,f_m)\mathcal{O}_0$. Moreover, one can write an explicit division formula

$$h(z) - T_0^{q,\rho}h(z) = \sum_{j=1}^{m} T_j^{q,\rho}h(z) f_j(z),\ z\in\widetilde{U}, \tag{2.5}$$

where the $T_j^{q,\rho}h$ are holomorphic functions in $\widetilde{U}$.

Proof. The proof of this result, when $q = 0$ and $\rho_j \equiv 1$ for any j is given in [DGSY, Section 5]. The method can be immediately extended to our case. It is based on the weighted Bochner-Martinelli formulas for division (see, for example, in [BGVY, Proposition 5.18], or Section 3 in Chapter 2 of the same reference). We will follow the notations used in the above references. We just need to express the Berndtsson-Andersson weighted representation formula with one weight (q, Γ), where

$$q(z,\zeta) = q_\lambda(z,\zeta) = \|f\|_{q,\rho}^{2(\lambda-1)} \Big(\sum_{j=1}^m s_j^{q,\rho,1} g_{j1}(\zeta,z), \dots, \sum_{j=1}^m s_j^{q,\rho,1} g_{jn}(\zeta,z) \Big) = (q_{\lambda,1}, \dots, q_{\lambda,n})$$

and $\Gamma(t) = t^m$, where λ is a complex parameter such that $\operatorname{Re}\lambda > 2$. We let

$$Q_\lambda(z,\zeta) = \sum_{k=1}^n q_{\lambda,k} d\zeta_k$$

and

$$\Sigma(z,\zeta) = \sum_{k=1}^n \sigma_k(z,\zeta) d\zeta_k \,.$$

If we write

$$\mathbf{K}_\lambda(z,\zeta) =$$
$$\sum_{l=0}^m \binom{m}{l} \Big(1 - \|f\|_{q,\rho}^2 + \|f\|^{2(\lambda-1)} < s^{q,\rho,1}, f(z) > \Big)^{m-l} \Big[\Sigma \wedge (\overline{\partial}_\zeta \Sigma)^{n-1-l} \wedge (\overline{\partial}_\zeta Q_\lambda)^l\Big] \,,$$

we have, for any z in $\widetilde{U}$,

$$h(z) = -\frac{1}{(2\pi i)^n} \int_U h(\zeta) d\varphi(\zeta) \wedge \mathbf{K}_\lambda(z,\zeta) \,. \tag{2.6}$$

We now consider (2.6) as an equality between two meromorphic functions of λ which have no pole at the origin. The identity

$$h(z) = -\frac{1}{(2\pi i)^n} \left[\int_U h(\zeta) d\varphi(\zeta) \wedge \mathbf{K}_\lambda(z,\zeta) \right]_{\lambda=0} \,,$$

together with the formulas (2.3) and the definition of our residual currents, gives the division formula (2.5). $\diamondsuit$

As an application of this theorem, we would like to mention the following result. When $f_1, \dots, f_n$ are n elements in ${}_n\mathcal{O}_0$ defining a regular sequence, it is a classical fact that the germ of the Jabobian $J = J(f_1, \dots, f_n)$ cannot be in the ideal $(f_1, \dots, f_n)\ {}_n\mathcal{O}_0$ (see for example [EiL]). In fact, one has

$$\dim \frac{{}_n\mathcal{O}_0}{(f_1, \dots, f_n)} = \operatorname{Res} \begin{bmatrix} J(\zeta) d\zeta_1 \wedge \cdots \wedge d\zeta_n \\ f_1, \dots, f_n \end{bmatrix} \,.$$

If the Jacobian were in the ideal $(f_1, \ldots, f_n)$, we would have have, from the local duality theorem, $\dim \frac{{}_n\mathcal{O}_0}{(f_1,\ldots,f_n)} = 0$, which is absurd. On the other hand, when $P_1, \ldots, P_n$ are homogeneous polynomials in n variables defining a non discrete variety (that is, the set of common zeroes contains other points besides the origin), it was claimed by E. Netto ([Net], vol 2, §441) and proved in [Sp] than the Jacobian of $P_1, \ldots, P_n$ lies in the ideal generated by the P_j, $j = 1, \ldots, n$. This problem was pointed to us by A. Ploski. Using our methods, we can prove the following local result.

Proposition 2.1. *Let $f_1, \ldots, f_n \in {}_n\mathcal{O}_0$, such that the germ of variety $V(f_1, \ldots, f_n)$ equals set theoretically the germ of variety of $V(f_1, \ldots, f_\nu)$ for some $\nu < n$. Then, the germ of the Jacobian $J = J(f_1, \ldots, f_n)$ is in the ideal $(f_1, \ldots, f_n)\,{}_n\mathcal{O}_0$. If one takes representatives f_j for the germs, the quotients T_jJ in the division formula*

$$J = \sum_{j=1}^{n} T_j J(z) f_j(z),\ z \in \widetilde{U}$$

(where $\widetilde{U}$ is a neighborhood of 0) can be expressed in terms of the action of currents that can be defined directly from the analytic continuation of $\lambda \mapsto F^\lambda$, where $F = |f_1|^2 + \cdots + |f_\nu|^2 + |f_{\nu+1}|^{2N} + \cdots + |f_n|^{2N}$ for some convenient $N \in \mathbf{N}^*$.

Proof. We will consider $f_1, \ldots, f_n$ as germs in ${}_{n+1}\mathcal{O}_0$ (depending only of the first n coordinates $\zeta_1, \ldots, \zeta_n$). We take representatives for the f_j, they define in some neighborhood U of the origin in $\mathbf{C}^{n+1}$ an analytic variety $V(f)$ with codimension strictly less than n, which is set theoretically the same as $V(f_1, \ldots, f_\nu)$. Let g_{jl}, $1 \le j, l \le n$ be any collection of holomorphic functions in $U \times U$, depending on $\zeta_1, \ldots, \zeta_n, z_1, \ldots, z_n$, such that

$$f_j(z) - f_j(\zeta) = \sum_{l=1}^{n} g_{jl}(z,\zeta)(z_l - \zeta_l),\ j = 1, \ldots, n\,.$$

Let φ a test function in $\mathcal{D}(\mathbf{C}^{n+1})$, with compact support in U, which is identically equal to 1 in a neighborhhood $\widetilde{U}$ of the origin. We know that near any point z_0 of $V(f_1, \ldots, f_n) = V(f_1, \ldots, f_\nu)$ in $\mathrm{supp}\,(d\varphi)$, the germs at z_0 of $f_{\nu+1}, \ldots, f_n$ are in the radical of the ideal $(f_1, \ldots, f_\nu)\,{}_{n+1}\mathcal{O}_{z_0}$. Local Lojasiewicz inequalities imply that there exists M such that in a neighborhood of $\mathrm{supp}\,(d\varphi)$, $f_{\nu+1}^M, \ldots, f_n^M$ are locally in the integral closure of the ideal generated by $(f_1, \ldots, f_\nu)$. We choose $\rho_j \equiv 1$, $j = 1, \ldots, n$, $q_j = 0$, $j = 1, \ldots, \nu$, $q_j = nM$, $j = \nu + 1, \ldots, n$. In order to prove the proposition, it is enough to prove (because of Theorem 2.1) that

$$\sum_{1\le r\le n} \sum_{\substack{i_1<\cdots<i_{n+1-r}\\ 1\le i_l\le n+1}} \sum_{\substack{j_1<\cdots<j_r\\ 1\le j_s\le n}} \left(\gamma_{n+1-r}\,\mathrm{Res}\begin{bmatrix} J d\varphi \wedge \Omega(\sigma(z,\zeta);\mathcal{I}) \wedge \big(\bigwedge_{l=1}^{n+1-r} d\zeta_{i_l}\big) \wedge \bigwedge_{s=1}^{r} G_{j_s}(z,\zeta) \\ f_{j_1}, \ldots, f_{j_r} \\ f_1, \ldots, f_n \end{bmatrix}^{q,\rho}\right) = 0 \tag{2.7}$$

for any $z \in U$, where σ is a $n+1$-valued function in (z,ζ), defined in $\widetilde{U} \times W$, W being a neighborhood of $\operatorname{supp}(d\varphi)$, and

$$d\varphi(\zeta) \neq 0 \Longrightarrow \sum_{k=1}^{n+1} \sigma_k(z,\zeta)(\zeta_k - z_k) = 1\,.$$

We first want to show that all the residue symbols in (2.7) corresponding to subsets $\mathcal{J} = \{j_1, \dots, j_r\} \subset \{1, \dots, n\}$ with cardinal strictly less than n are identically zero (as functions of z). We first notice that if $\mathcal{J}$ is such a ordered subset of $\{1, \dots, n\}$, with cardinal $r < n$, and $\mathcal{I} = \{i_1, \dots, i_{n+1-r}\}$ is any ordered subset of $\{1, \dots, n+1\}$ with cardinal $n+1-r$, we have

$$\Big(\prod_{s=1}^{r} f_{j_s}^{q_{i_s}}\Big) J\, d\zeta_1 \wedge \ldots \wedge d\zeta_n = \Big(\prod_{s=1}^{r} f_{j_s}^{q_{j_s}+1}\Big)\Big(\bigwedge_{s=1}^{r} \frac{df_{j_s}}{f_{j_s}}\Big) \wedge \bigwedge_{j \notin \mathcal{J}} df_j \tag{2.8}$$

and

$$J d\varphi \wedge \Omega(\sigma(z,\zeta);\mathcal{I}) \wedge \big(\bigwedge_{l=1}^{n+1-r} d\zeta_{i_l}\big) \wedge \bigwedge_{s=1}^{r} G_{j_s}(z,\zeta) = J d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge \phi$$

where ϕ is a $(1,r)$-differential form with smooth coefficients of compact support in U. As in Section 1, let

$$\Theta_\lambda = \lambda \|f\|_{q,\rho}^{2(\lambda - r)} \overline{\partial} \|f\|_{q,\rho}^2 \wedge \Omega(s^{q,\rho,1};\mathcal{J}),$$

where λ is a complex parameter. Let z_0 be a common zero of $(f_1, \dots, f_n)$ in the support of $d\varphi$ and $\pi : \mathcal{X}_{z_0} \mapsto W(z_0)$ a resolution of singularities near z_0 for $\{f_1 \cdots f_n = 0\}$, such that in local coordinates on $\mathcal{X}_{z_0}$ (centered at a point x), one has, in the corresponding local chart U_x around x,

$$(f_j \circ \pi(t))^{q_j+1} = u_j(t) t_1^{\alpha_{j,1}} \cdots t_{n+1}^{\alpha_{j,n+1}} = \theta_j(t) t^{\alpha_j},\ j = 1, \dots, n,$$

where the u_j, $j = 1, \dots, n$, are non vanishing holomorphic functions and at least one of the monomials $t^{(q_j+1)\alpha_j} = \mu(t)$, $j = 1, \dots, n$, divides any $t^{(q_k+1)\alpha_k}$, $k = 1, \dots, n$. Recall that the function

$$\lambda \mapsto J^{q,\rho}\Big(J d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge \phi; \mathcal{J}; \lambda\Big)$$

is a meromorphic function of λ such that

$$\begin{aligned} &J^{q,\rho}(J d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge \phi; \mathcal{J}; 0) = \\ &= \operatorname{Res} \begin{bmatrix} J d\varphi \wedge \Omega(\sigma(z,\zeta);\mathcal{I}) \wedge \big(\bigwedge_{l=1}^{n+1-r} d\zeta_{i_l}\big) \wedge \bigwedge_{s=1}^{r} G_{j_s}(z,\zeta) \\ f_{j_1}, \dots, f_{j_r} \\ f_1, \dots, f_n \end{bmatrix}^{q,\rho}. \end{aligned}$$

This function of λ is a combination of terms of the form

$$\int_\Omega \pi^* \Theta_\lambda \wedge \psi \pi^*(J d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge \phi), \tag{2.9}$$

where ψ is a member of a partition of unity for $\pi^*(\mathrm{supp}(d\varphi))$. If we compute $\pi^*\Theta_\lambda$ (using (1.4) and (2.8)), we can express (2.9) as

$$\lambda \int_\Omega |a\mu|^{2\lambda}\Big(\tilde{\vartheta} + \tilde{\varpi} \wedge \frac{\overline{\partial\mu}}{\overline{\mu}}\Big) \wedge \Big(\bigwedge_{s=1}^{r} \frac{d(\pi^* f_{j_s})}{\pi^* f_{j_s}}\Big) \wedge \bigwedge_{j\notin\mathcal{J}} d(\pi^* f_j) \wedge \psi\pi^*\phi\,,$$

where $\tilde{\vartheta}$ and $\tilde{\varpi}$ are smooth differential forms of respective types $(0,r)$, $(0,r-1)$, and a is a non vanishing function. Suppose now that t_ι is a coordinate that divides μ; then, it divides all $\pi^* f_j$, $j=1,\dots,n$. For any $j\in\{1,\dots,n\}$, in particular, when $j\notin\mathcal{J}$, we have

$$\pi^*(df_j) = d(\pi^* f_j) = t_\iota\xi_1 + \xi_2 dt_\iota,$$

where ξ_1 and ξ_2 are $(0,1)$ and $(0,0)$ forms in U_x. Therefore, since

$$\bigwedge_{s=1}^{r} \frac{d(\pi^* f_{j_s})}{\pi^* f_{j_s}}$$

is a wedge product of logarithmic derivatives, the differential form

$$\Big(\bigwedge_{s=1}^{r} \frac{d(\pi^* f_{j_s})}{\pi^* f_{j_s}}\Big) \wedge \bigwedge_{j\notin\mathcal{J}} d(\pi^* f_j)$$

does not have t_ι as a factor in its denominator. But the only possible holomorphic non vanishing factors in the denominator of

$$\pi^*\Theta_\lambda \wedge \psi\pi^*(J d\zeta_1\wedge\cdots\wedge d\zeta_n\wedge\phi)$$

are of the form $t_\iota^{k_\iota}$, since we have from (1.4)

$$\pi^*\Theta_\lambda = \lambda\frac{|a\mu|^{2\lambda}}{\mu^r}\Big(\prod_{s=1}^{r}(\pi^* f_{j_s})^{q_{j_s}}\Big)\Big(\vartheta + \varpi\wedge\frac{\overline{\partial\mu}}{\overline{\mu}}\Big)\,,$$

where ϑ and ϖ are smooth differential forms of type $(0,r)$, $(0,r-1)$ respectively (see (1.4)). This means that the differential form

$$\pi^*\Theta_\lambda \wedge \psi\pi^*(J d\zeta_1\wedge\cdots\wedge d\zeta_n\wedge\phi)$$

has no holomorphic singularities. We conclude that $(Jd\zeta_1\wedge\cdots\wedge d\zeta_n\wedge\phi;\mathcal{J};0)=0$, which means that

$$\mathrm{Res}\begin{bmatrix} Jd\varphi\wedge\Omega(\sigma(z,\zeta);\mathcal{I})\wedge\big(\bigwedge_{l=1}^{n+1-r} d\zeta_{i_l}\big)\wedge\bigwedge_{s=1}^{r} G_{j_s}(z,\zeta) \\ f_{j_1},\dots,f_{j_r} \\ f_1,\dots,f_n \end{bmatrix}^{q,\rho} = 0\,.$$

It remains for us to show that, for any $z \in U$,

$$\mathrm{Res}\begin{bmatrix} J\sigma_{n+1} d\varphi \wedge d\zeta_{n+1} \wedge \bigwedge_{j=1}^{n} G_j(z,\zeta) \\ f_1, \dots, f_n \\ f_1, \dots, f_n \end{bmatrix}^{q,\rho} = 0 \,. \tag{2.10}$$

We know also that if U is small enough, which we can always assume, the radical of $(f_1, \dots, f_n)$ is the radical of $(f_1, \dots, f_\nu)$. Let us consider again a point z_0 in $V(f) = V(f_1, \dots, f_\nu) \cap \mathrm{supp}\,(d\varphi)$; in a neighborhood of such point, $f_{\nu+1}, \dots, f_n$ are identically zero on any component of the analytic set $\{f_1 = \dots = f_\nu = 0\}$ that contains z_0. Let as before $\pi : \mathcal{X}_{z_0} \mapsto W(z_0)$ (where $W(z_0)$ is a neighborhhood of z_0) be a resolution of singularities such that in local coordinates on $\mathcal{X}_{z_0}$ (centered at a point x), one has, in the corresponding local chart U_x around x,

$$f_j \circ \pi(t) = u_j(t) t_1^{\alpha_{j,1}} \cdots t_{n+1}^{\alpha_{j,n+1}} = u_j(t) t^{\alpha_j},\ j = 1, \dots, \nu,$$

where the u_j are non vanishing holomorphic functions and at least one of the monomials $t^{\alpha_j} = \mu(t)$, $j = 1, \dots, d$, divides any t^{α_k}, $k = 1, \dots, \nu$. As before, it divides also any $\pi^* f_j^{q_j+1}$, $j = 1, \dots, n$, because $q_j = nM > M$ for $j = \nu+1, \dots, n$. We even know that μ^n divides $\pi^* f_{\nu+1}^{nM}, \dots, \pi^* f_n^{nM}$, since any f_j^{nM}, $j = \nu+1, \dots, n$, is in the n-th power of the integral closure of the ideal generated by the germs of $f_1, \dots, f_\nu$ in ${}_{n+1}\mathcal{O}_{z_0}$. We can write

$$\pi_* \|f\|_{q,\rho}^2 = |a\mu|^2 + \sum_{j=\nu+1}^{n} \pi^* |f_j|^{2nM} = |\tilde{a}\mu|^2,$$

where a and $\tilde{a}$ are non vanishing functions in the local chart. Therefore, if we set

$$\Theta_\lambda = \lambda \|f\|_{q,\rho}^{2(\lambda-n)} \overline{\partial} \|f\|_{q,\rho}^2 \wedge \Omega(s^{q,\rho,1}; \{1, \dots, n\}),$$

we have, in local coordinates in the local chart,

$$\pi^* \Theta_\lambda = \lambda \frac{|a\mu|^{2\lambda}}{\mu^n} \Big(\prod_{j=\nu+1}^{n} \pi^* f_j \Big)^{nM} \Big(\vartheta + \varpi \wedge \frac{\overline{\partial\mu}}{\overline{\mu}} \Big) \,. \tag{2.11}$$

The factor $\Big(\prod_{j=\nu+1}^{n} \pi^* f_j \Big)^{nM}$ in (2.11) compensates the singularity in μ^n. Thus, the differential form (2.11) has only antiholomorphic singularities. Now, since

$$\lambda \mapsto J^{q,\rho} \Big(J\sigma_{n+1} d\varphi \wedge d\zeta_{n+1} \wedge \bigwedge_{j=1}^{n} G_j(z,\zeta); \{1, \dots, n\}; \lambda \Big)$$

is a combination of integrals of the form

$$\int_{U_x} \pi^* \Theta_\lambda \wedge \psi \pi^* \Big(J\sigma_{n+1} d\varphi \wedge d\zeta_{n+1} \wedge \bigwedge_{j=1}^{n} G_j(z,\zeta) \Big)$$

for $x \in \mathcal{X}_{z_0}$, we have

$$J^{q,\rho}\Big(J\sigma_{n+1}d\varphi \wedge d\zeta_{n+1} \wedge \bigwedge_{j=1}^{n} G_j(z,\zeta);\{1,\dots,n\};0\Big) = \mathrm{Res}\begin{bmatrix} J\phi \\ f_1,\dots,f_n \\ f_1,\dots,f_n \end{bmatrix}^{q,\rho} = 0$$

and the proof of our proposition is complete. Note that, as a consequence of Theorem 2.1, we have also in this case an explicit division formula

$$J(z) = \sum_{j=1}^{n} T_j J(z) f_j(z),\ z \in \widetilde{U}\,. \qquad \diamondsuit$$

Remark 2.1. In fact, the only terms for which we had to introduce the weight q and use the geometric hypothesis on $V(f)$ are the terms of the form (2.10). In general, one has

$$\begin{aligned} T_0J(z) &= -\frac{1}{(2\pi i)}\mathrm{Res}\begin{bmatrix} J\sigma_{n+1}d\varphi \wedge d\zeta_{n+1} \wedge \bigwedge_{j=1}^{n} G_j(z,\zeta) \\ f_1,\dots,f_n \\ f_1,\dots,f_n \end{bmatrix}^{q,\rho} \\ &= -\frac{1}{2i\pi(nM+1)^{n-\nu}}[f]_n^{q,\rho}\Big(\det[g_{jl}(z,\zeta)]\sigma_{n+1}(z,\zeta)\overline{\partial}\varphi \wedge d\zeta_{n+1}\Big),\ z \in \widetilde{U}\,, \end{aligned}$$

and

$$(J - T_0J)_0 \in (f_1,\dots,f_n)\,{}_n\mathcal{O}_0\,.$$

Since the (n,n) current $[f]_n^{q,\rho}$ is positive, and therefore is of the form

$$\Big(\frac{1}{2i}\Big)^n \Theta \bigwedge_{l=1}^{m} d\overline{\zeta_l} \wedge d\zeta_l\,,$$

where Θ is a positive measure, then, for any holomorphic function h in U which vanishes on $V(f)$, one has $T_0(hJ) = 0$, which means that hJ is locally in $\widetilde{U}$ in the ideal generated by $(f_1,\dots,f_n)$. This result is well known when $f_1,\dots,f_n$ define the origin as an isolated zero (it follows from Kronecker's interpolation formula [GH]).

In fact, we have the following theorem.

Theorem 2.2. *Let $f_1,\dots,f_n$ be n germs of holomorphic functions in ${}_n\mathcal{O}_0$ which define an ideal with analytic spread ν strictly less than n. Then, the germ at 0 of the Jacobian $J = J(f_1,\dots,f_n)$ is in the ideal $(f_1,\dots,f_n)\,{}_n\mathcal{O}_0$.*

Proof. Consider $\tilde{f}_1,\dots,\tilde{f}_\nu$ such that the germs at 0 of $(\tilde{f}_1,\dots,\tilde{f}_\nu)$ define an ideal with the same integral closure than the ideal generated by the germs of the f_j. As before, we take representatives for the germs in some neighborhood U of the origin in $\mathbf{C}^n$. and functions holomorphic $\tilde{g}_{jk}$ in $U \times U$ such that

$$\tilde{f}_j(z) - \tilde{f}_j(\zeta) = \sum_{k=1}^{n} \tilde{g}_{jk}(z,\zeta)(z_k - \zeta_k),\ j = 1,\dots,\nu\,.$$

We consider a test function φ with support in U, which is identically zero in some neigborhood $\widetilde{U}$ of the origin and a n-complex valued function σ of $2n$ variables (z,ζ), defined in $\widetilde{U}\times W$, where W is a neighborhood of the support of $d\varphi$, holomorphic in z, C^1 in ζ such that

$$d\varphi(\zeta)\neq 0 \Longrightarrow \sum_{k=1}^{n}\sigma_k(z,\zeta)(\zeta_k-z_k)\,.$$

In order to prove that J belongs to the ideal $(f_1,\dots,f_n)$, it is enough to prove that J belongs to the ideal $(\tilde{f}_1,\dots,\tilde{f}_\nu)$.From Theorem 2.1, it is enough to show that for any $z\in U$,

$$\sum_{1\leq r\leq\nu}\ \sum_{\substack{i_1<\dots<i_{n-r}\\ 1\leq i_l\leq n}}\ \sum_{\substack{j_1<\dots<j_r\\ 1\leq j_s\leq t}}$$

$$\left(\gamma_{n-r}\,\mathrm{Res}\begin{bmatrix} Jd\varphi\wedge\Omega(\sigma(z,\zeta);\mathcal{I})\wedge\big(\bigwedge_{l=1}^{n-r}d\zeta_{i_l}\big)\wedge\bigwedge_{s=1}^{r}\widetilde{G}_{j_s}(z,\zeta)\\ \tilde{f}_{j_1},\dots,\tilde{f}_{j_r}\\ \tilde{f}_1,\dots,\tilde{f}_\nu\end{bmatrix}^{q,\rho}\right)=0\,,$$

where we take here $q=(q_1,\dots,q_\nu)=(0,\dots,0)$ and $\rho=(\rho_1,\dots,\rho_\nu)\equiv(1,\dots,1)$. As before, we consider, for any point in $V(\tilde{f})=V(f)$, a desingularization $\pi_{z_0}:\mathcal{X}_{z_0}\mapsto W(z_0)$, such that in local coordinates on $\mathcal{X}_{z_0}$ (centered at a point x), one has, in the corresponding local chart U_x around x,

$$\tilde{f}_j\circ\pi(t)=u_j(t)t_1^{\alpha_{j,1}}\cdots t_n^{\alpha_{j,n}}=u_j(t)t^{\alpha_j},\ j=1,\dots,\nu,$$

where the u_j are non vanishing holomorphic functions and at least one of the monomials $t^{\alpha_j}=\mu(t)$, $j=1,\dots,\nu$, divides any t^{α_k}, $k=1,\dots,\nu$. Since the f_j are in the integral closure of the ideal defined by the $\tilde{f}_j$, μ divides any π^*f_j, $j=1,\dots,n$. It follows from that that μ^{n-1} divides $\pi^*(df_1)\wedge\cdots\wedge\pi^*(df_n)$. Then, for any $r\in\{1,\dots,\nu\}$, for any subset $\mathcal{J}$ of $\{1,\dots,\nu\}$ with cardinal r, the differential form

$$\lambda\pi^*\Big[\|f\|_{q,\rho}^{2(\lambda-r)}\overline{\partial}\|f\|_{q,\rho}^2\wedge\Omega(s^{q,\rho,1};\mathcal{J})\Big]\wedge\bigwedge_{j=1}^{n}\pi^*(df_j)$$

has no holomorphic singularities. This implies that, for any such $\mathcal{J}$, for any $\mathcal{I}\subset\{1,\dots,n\}$, $\#\mathcal{I}=n-r$, for any $z\in\widetilde{U}$, one has

$$\mathrm{Res}\begin{bmatrix} Jd\varphi\wedge\Omega(\sigma(z,\zeta);\mathcal{I})\wedge\big(\bigwedge_{l=1}^{n-r}d\zeta_{i_l}\big)\wedge\bigwedge_{s=1}^{r}\widetilde{G}_{j_s}(z,\zeta)\\ \tilde{f}_{j_1},\dots,\tilde{f}_{j_r}\\ \tilde{f}_1,\dots,\tilde{f}_\nu\end{bmatrix}^{q,\rho}=0$$

(it is enough to look at the behavior near 0 of the meromorphic function of λ whose value at 0 is precisely this residue symbol). This completes the proof of the theorem. $\diamondsuit$

These results can also be stated from the global point of view. For example, we have the following theorem, extending partially Netto's statement to the affine case.

Theorem 2.3. *Let $P_1, \dots P_n$ be n polynomials in n variables such that the zero set of $P_1, \dots, P_n$ can be defined as the zero set of $P_1, \dots, P_\nu$, with $\nu < n$. Then, the Jacobian $J(P_1, \dots, P_n)$ of $(P_1, \dots, P_n)$ is in the ideal generated by the P_j, $1 \le j \le n$. Moreover, one has a division formula*

$$J = A_1 P_1 + \cdots + A_n P_n,$$

where the A_j can be computed in terms of the analytic continuation of the map

$$\lambda \mapsto \Big(|P_1|^2 + \cdots + |P_\nu|^2 + |P_{\nu+1}|^{2(nN+1)} + \cdots + |P_n|^{2(nN+1)} \Big)^{\lambda},$$

where N is such that

$$(\text{rad } (P_1, \dots, P_\nu))^N \subset \text{local integral closure of } (P_1, \dots, P_\nu)\,.$$

Remark. Using local Lojasiewicz inequalities ([JKS], [Cyg]) and the Briançon-Skoda theorem [BS], one can choose $N = \prod_{k=1}^{\nu} D_k$.

Proof. We use the weighted Bochner-Martinelli formulas with two pairs of weights (Q_λ, t^n) and $(\overline{\partial}\partial \log(1 + \|\zeta\|^2), t^M)$ for M large enough and

$$Q_\lambda = \sum_{k=1}^{n} q_{\lambda,k}(z,\zeta) d\zeta_k,$$

where

$$q_{\lambda,k} = \|P\|_N^{2(\lambda-1)} \Big(\sum_{j=1}^{\nu} \overline{P_j} g_{jk}(z,\zeta) + \sum_{j=\nu+1}^{n} \overline{P_j} |P_j|^{2nN} g_{jk}(z,\zeta) \Big),$$

with

$$\|P\|_N^2 = \sum_{k=1}^{\nu} |P_k|^2 + \sum_{k=\nu+1}^{n} |P_k|^{2(nN+1)},$$

and the g_{jk} satisfying

$$P_j(z) - P_j(\zeta) = \sum_{k=1}^{n} g_{jk}(z,\zeta)(z_k - \zeta_k), \ j = 1, \dots, n\,.$$

Let K_λ and P_λ be the two kernels involved in the representation formulas (we refer to [BGVY] for the details and the notations). Then, if φ is a test function identically equal to 1 in some neighborhood u of the origin and $R > 0$, one has, for any $z \in u$,

$$J(z) = \frac{1}{(2\pi i)^n} \bigg(\int J(\zeta) \varphi(\frac{\zeta}{R}) P_\lambda(z,\zeta) - \frac{1}{R} \int J(\zeta) \overline{\partial} \varphi(\frac{\zeta}{R}) \wedge K_\lambda(z,\zeta) \bigg)\,. \qquad (2.12)$$

We consider (2.12), when R is fixed, as an identity between two meromorphic functions of λ, then let $\lambda = 0$ by following the analytic continuation, and finally let R tend to infinity.

The choice of N is made possible by the control one has on the growth of the distributions (of the principal value type or coefficients of residue currents) involved as coefficients in the Laurent developments at its poles of the meromorphic function

$$\lambda \mapsto \|P\|_N^{2\lambda}$$

(see for example [BY1], Proposition 5). ♢

3. Green currents and purely dimensional cycles.

In this section, we shall give another application of the same ideas. We will explain how to construct a Green current G relative to a purely dimensional effective cycle Z in $\mathbf{P}^n(\mathbf{C})$ which can be decomposed into irreducible ones as

$$Z = \sum_{i=1}^{s} m_i Z_i,\ m_i \in \mathbf{N}^*,\ \operatorname{codim}(Z_i) = d,\ i = 1, \ldots, s,$$

in terms of global sections $P_1, \ldots, P_m$, that generate the ideal sheaf

$$I(Z) = \sum_{i=1}^{s} I(Z_i)^{m_i},$$

where $I(Z_i)$ denotes the ideal sheaf of Z_i. Here $P_1, \ldots, P_m$ are homogeneous polynomials in $n+1$ variables with respective degrees $D_1 \geq D_2 \geq \cdots \geq D_m$. More precisely, we would like to construct a $(d-1, d-1)$ current $\mathbf{G}_Z$ such that

$$dd^c \mathbf{G}_Z + (\deg Z)\, \omega^p = \delta_Z = \sum_{i=1}^{s} m_i \deg I(Z_i)\, \delta_{[Z_i]},$$

where $\omega = dd^c \log(|x_0|^2 + \cdots + |x_n|^2)$ defines the Kahler metric on $\mathbf{P}^n(\mathbf{C})$ and $\delta_{[Z_i]}$ denotes the integration current (without multiplicities) on the reduced algebraic variety $V(I([Z_i]))$. Moreover, we would like $\mathbf{G}_Z$ to be smooth outside the support of the cycle Z. (So that, later on, we can use such a current to express in terms of the polynomials $P_1, \ldots, P_m$, the analytic contribution to the arithmetic height of Z, whenever the P_j are in $\mathbf{Z}[x_0, \ldots, x_n]$.) Such a construction was done in [BY] under the condition that $I([Z]) = (P_1, \ldots, P_d)$, that is the cycle Z is defined as a complete intersection (or the divisors $\{P_j = 0\}$, $j = 1, \ldots, d$, intersect properly). Our construction will be based on the following theorem.

Theorem 3.1. *Let $P_1, \ldots, P_m$, be m homogeneous polynomials in $n+1$ variables, with respective degrees $D_1 \geq \ldots \geq D_m$, defining a purely $n-d$-dimensional algebraic variety $V(P)$ in $\mathbf{P}^n(\mathbf{C})$, and Z be the cycle associated to the ideal sheaf $(P_1, \ldots, P_m)\mathcal{O}_{\mathbf{P}^n(\mathbf{C})}$. Then, for $N \geq dD_1^d$ and for generic complex values β_{jk}, $j = 1, \ldots, d$, $k = 1, \ldots, m$,*

$\beta_{0l},\ l=0,\ldots,n$, *the meromorphic current-valued map (with values in the space of* (d,d) *currents in* $\mathbf{P}^n(\mathbf{C})$*) defined by*

$$\lambda \mapsto I_\lambda =$$

$$\frac{\lambda(d-1)!}{(2i\pi)^d}\|Q\|_{\rho,q}^{2(\lambda-p-1)}\overline{\partial}\|Q\|^2_{q,\rho}\wedge\partial\|Q\|^2_{q,\rho}\wedge\sum_{\substack{j_1<\cdots<j_{d-1}\\ 1\le j_r\le m+d}}\bigwedge_{l=1}^{d-1}\overline{\partial}(\rho_{j_l}\overline{Q_{j_l}}^{q_{j_l}+1})\wedge\partial(\rho_{j_l}Q_{j_l}{}^{q_{j_l}+1}), \tag{3.1}$$

where

$$\begin{cases} q_j = 0,\ j=1,...,d \\ \rho_j = \|x\|^{-D_1},\ j=1,\ldots,d \end{cases} \qquad \begin{cases} q_j = N,\ j=d+1,\ldots,m+d \\ \rho_j = \|x\|^{-(N+1)D_j},\ \ j=d+1,\ldots,d+m \end{cases}$$

$$Q_j = \sum_{k=1}^m \beta_{jk}\big(\sum_{l=0}^n \beta_{0l}x_l\big)^{D_1-D_k}P_k,\ j=1,\ldots,d,$$

$$Q_j = P_{j-d},\ \ j=d+1,\ldots,d+m,$$

$$\|Q\|^2_{q,\rho} = \sum_{j=1}^{m+d}\rho_j^2|Q_j|^{2(q_j+1)},$$

is holomorphic at $\lambda=0$ *and such that* I_0 *is the integration current (with multiplicities)* δ_Z.

Proof. If the P_j define a discrete variety in $\mathbf{P}^n(\mathbf{C})$, then we choose the coefficients β_{0l}, $l=0,\ldots,n$, such that the hyperplane $\Gamma=\{\sum_{l=0}^n\beta_{0l}x_l=0\}$ does not intersect the support of the cycle Z. If the P_j define a variety with codimension $1\le d<n$, then, we choose the β_{0l} such that the hyperplane $\{\sum_{l=0}^n\beta_{0l}x_l=0\}$ intersects properly any connected component of $\mathrm{Reg}\,(V(P))$, where $\mathrm{Reg}\,(V(P))$ is the set of regular points in $V(P)$. We will denote by Λ the linear form

$$\Lambda(x)=\sum_{l=0}^n\beta_{0l}x_l\,.$$

Let $\Gamma_1,\ldots,\Gamma_T$ the different connected components of $\mathrm{Reg}\,(V(P))\setminus\Gamma$, and x_τ, $1\le\tau\le T$, a generic point in Γ_τ. In the discrete case, the points x_τ, $\tau=1,\ldots,T$, will be by definition the points in $V(P)$.

We claim that, when $d<n$, one can choose the generic point x_τ on Γ_τ such that if λ_{jk}, $j=1,\ldots,d$, $k=1,\ldots,m$, are generic complex coefficients, then the polynomials $(P_1,\ldots,P_m)$ and the polynomials

$$Q_{\lambda,j}(x)=\sum_{k=1}^m\lambda_{jk}\Lambda(x)^{D_1-D_k}P_k(x),\ j=1,\ldots,d,$$

define the same (smooth) algebraic variety in a neighborhood of x_τ. In order to see that, we proceed as follows. Let $\mathbf{F}$ be an algebraic closure of the field $\mathbf{C}(\lambda_{jk};\ 1\le j\le d;\ 1\le k\le m)$.

We consider the polynomials $Q_{\lambda,j}$ as homogeneous polynomials with coefficients in $\mathbf{F}$ and the primary decomposition

$$(Q_{\lambda,1},\ldots,Q_{\lambda,d}) = \bigcap_{\iota} \mathcal{P}_{\iota}$$

in the polynomial ring $\overline{\mathbf{F}}[x]$. We consider only the isolated primes $\mathcal{P}_\iota$ in this decomposition whose zero set contains x_τ. Among them, there is the prime ideal $\mathcal{P}$ which defines the smooth algebraic set $V(P)$ near x_τ. If $\mathcal{P}_\iota$ is different from $\mathcal{P}$, the zero variety (in $\mathbf{P}^n(\overline{\mathbf{F}})$) of $\mathcal{P}_\iota$ intersects $V(P)$ (near τ in $\mathbf{P}^n(\overline{\mathbf{F}})$) along a variety with dimension strictly less that $n-d$. This implies that one can choose $\tilde{x}_\tau$ close to x_τ on Γ_τ and such that $\tilde{x}_\tau$ is not in any of the zero sets $V(\mathcal{P}_\iota) \subset \mathbf{P}^n(\overline{\mathbf{F}})$, where $\mathcal{P}_\iota \neq \mathcal{P}$. This means that for generic values of λ, for any such ι, $\tilde{x}_\tau$ is not a common zero of the polynomials $x \mapsto p_{\iota,l}(\lambda,x)$, where the $p_{\iota,l}$ generate $\mathcal{P}_\iota$. We will choose this new point $\tilde{x}_\tau$ instead of x_τ. It is clear that at this new point x_τ, the polynomials $Q_{\lambda,1},\ldots,Q_{\lambda,d}$, define also $V(P)$ as a smooth variety near x_τ for any generic choice of the parameters λ.

Let $p_1,\ldots,p_m$, be the homogeneous polynomials P_j expressed in affine coordinates in some neighborhood of x_τ. Recall (see for example [Te], corollaire 5.4) that the multiplicity of $(p_1,\ldots,p_m)\,{}_n\mathcal{O}_{x_\tau}$ at x_τ equals the multiplicity of $(p_1,\ldots,p_m,L_{\tau,1},\ldots,L_{\tau,n-d})\,{}_n\mathcal{O}_{x_\tau}$, where $L_{\tau,1},\ldots,L_{\tau,n-d}$ are generic linear forms (expressed in affine coordinates) vanishing at x_τ. Let f_j, $j=1,\ldots,m$, be the germs at x_τ of the polynomials $P_j\Lambda^{D_1-D_k}$, $j=1,\ldots,m$, expressed in local coordinates (centered at x_τ). Recall that the f_j, $j=1,\ldots,m$, define in ${}_n\mathcal{O}_{x_\tau}$ the same ideal as the p_j, $j=1,\ldots,m$, since x_τ does not belong to the hyperplane Γ. Thus, the multiplicity at x_τ of

$$(P_1,\ldots,P_m,L_{\tau,1},\ldots,L_{\tau,n-d})\,{}_n\mathcal{O}_{x_\tau}$$

is also the multiplicity in $(\mathbf{C}^d,0)$ of the germ (in $(\mathbf{C}^d,0)$) of the map

$$t \mapsto (f_1(x_\tau + A_\tau t),\ldots,f_m(x_\tau + A_\tau t)),$$

where A_τ is a (n,d) matrix with generic coefficients (generic depends of course of the choice of x_τ). If we take d generic linear combinations (still depending on τ) of the germs $t \mapsto f_j(x_\tau + A_\tau t)$, we preserve the local multiplicity at x_τ, since the integral closure of the ${}_d\mathcal{M}_0$-primary ideal generated in ${}_d\,\mathcal{O}_0$ by these germs is the same than the integral closure in this local ring of the ideal generated by the $f_j(x_\tau + A_\tau t)$, $j=1,\ldots,m$ [NR]. Moreover, as we have seen above, we can choose these d generic linear combinations so that they define a smooth complete intersection near the point x_τ. Thus, if the β_{jk}, $j=1,\ldots,d$, $k=1,\ldots m$, are generic complex numbers, the multiplicity at any x_τ, $\tau=1,\ldots,T$, of the ideal generated by the P_j in $\mathcal{O}_{x_\tau}$ equals the multiplicity of the ideal generated by the germs at x_τ of the homogeneous polynomials Q_j, $j=1,\ldots,d$, where

$$Q_j(x) = \sum_{k=1}^{m} \beta_{jk}\Lambda(x)^{D_1-D_k}P_k(x),\ j=1,\ldots,d\,.$$

This local multiplicity remains constant on the whole connected component Γ_τ (we will denote it as m_τ). Moreover, the smooth complete intersection $\{Q_1=\ldots=Q_d=0\}$ is defined near x_τ as the zero set of some primary component $\mathcal{P}_\tau$ of the homogeneous ideal $(Q_1,\dots,Q_d)$. We will denote $\widetilde{\Gamma}_\tau = \Gamma_\tau \setminus \mathrm{Sing}\,(V(Q_1,\dots,Q_d))$. All points in $\widetilde{\Gamma}_\tau$ are smooth points both for Z and for the algebraic variety $V(Q_1,\dots,Q_d)$. At all these points, m_τ is also the local multiplicity of the ideal defined by the germs of the Q_j, $j=1,\dots,d$.

It is clear that, for any value of the complex parameter λ with large real part, the differential form in homogenous coordinates that appears in (3.1) defines a differential form in $\mathbf{P}^n(\mathbf{C})$. If φ is an $(n-d,n-d)$ test form in $\mathbf{P}^n(\mathbf{C})$, then $\int_{\mathbf{P}^n(\mathbf{C})} I_\lambda \wedge \varphi$ is the Mellin transform of the function

$$\epsilon \mapsto \Phi(\varphi;\epsilon) =$$
$$\frac{(d-1)!}{(2i\pi\epsilon)^d}\int_{\|Q\|^2_{\rho,q}=\epsilon} \partial\|Q\|^2_{q,\rho} \wedge \sum_{\substack{j_1<\cdots<j_{d-1}\\ 1\le j_r\le m+d}} \bigwedge_{l=1}^{d-1} \overline{\partial}(\rho_{j_l}\overline{Q_{j_l}}^{q_{j_l}+1}) \wedge \partial(\rho_{j_l} Q_{j_l}{}^{q_{j_l}+1}). \tag{3.2}$$

We know from Lemmas 1.1 and 1.2 that this last function has a limit when $\epsilon \to 0$. This limit equals $< [Q]^{q,\rho}_d, \varphi >$, where $[Q]^{q,\rho}_d$ is a closed positive current supported by $V(Q)=V(P)$. It follows that $\lambda \mapsto I_\lambda$ can be continued as a (d,d) current-valued meromorphic function with no pole at the origin, and the value I_0 at the origin is exactly the current $[Q]^{q,\rho}_d$. In order to conclude the proof of the theorem, we have to distinguish the cases $d=n$ and $d<n$. In the first case, we need to prove that the mass of the current $[Q]^{q,\rho}_d$ equals the multiplicity of Z at any point of the discrete variety $V(P)$. In the second case, it is enough to prove that our current coincides with the integration current (with multiplicities), near any point z_0 in each $\widetilde{\Gamma}_\tau$, $\tau=1,\dots,t$, since the union of these sets is dense in $\mathrm{Reg}\,(V(P))$, thus also in $V(P)$. Since the currents δ_Z and $[Q]^{q,\rho}_d$ are positive, closed, of type (d,d), and supported by the variety $V(P)$ of pure codimension d, they will concide. Therefore, we have to prove the two previous claims to conclude the proof. Since these claims are local, we can express the differential forms in affine coordinates in the local chart around z_0 in which we are working. Hence, in what follows we consider only the affine situation.

We have seen in the proof of Lemma 1.2 that both $\int_{\mathbf{P}^n(\mathbf{C})} I_\lambda \wedge \varphi$ and the Mellin transform of the following function

$$\widetilde{\Phi}(\varphi;\epsilon) =$$
$$\frac{\gamma_d}{\epsilon^d}\int_{\|Q\|^2_{q,\rho}=\epsilon} \Big[\sum_{\substack{i_1<\dots<i_d\\ 1\le i_l\le d+m}} \Big(\prod_{l=1}^{d}(q_{i_l}+1)\Big)\Omega(s^{q,\rho,1};\{i_1,\dots,i_d\}) \wedge \bigwedge_{l=1}^{d} dQ_{i_l}\Big] \wedge \varphi$$

(where $\gamma_d = \frac{(-1)^{\frac{d(d-1)}{2}}(d-1)!}{(2\pi i)^d}$ and $s^{q,\rho,1}_j = \rho_j^2 |Q_j|^{2q_j}\overline{Q_j}$ for $j=1,\dots,d+m$) take the same value at $\lambda=0$. We consider this function as a sum of the following two terms. The first one is

$$\widetilde{\Phi}_1(\varphi;\epsilon) = \frac{\gamma_d}{\epsilon^d}\int_{\|Q\|^2_{q,\rho}=\epsilon} \Omega(s^{q,\rho,1};\{1,\dots,d\}) \wedge dQ_1\wedge\cdots\wedge dQ_d \wedge \varphi. \tag{3.3}$$

The second one is

$$\widetilde{\Phi}_2(\varphi;\epsilon) =$$

$$\frac{\gamma_d}{\epsilon^d}\int_{\|Q\|^2_{q,\rho}=\epsilon}\Big[\sum_{\substack{i_1<\ldots<i_d\\ 1\le i_l\le d+m\\ \mathcal{I}\neq\{1,\ldots,d\}}}\Big(\prod_{l=1}^{d}(q_{i_l}+1)\Big)\Omega(s^{q,\rho,1};\{i_1,\ldots,i_d\})\wedge\bigwedge_{l=1}^{d}dQ_{i_l}\Big]\wedge\varphi\,. \tag{3.4}$$

The Mellin transform of the function $\lambda\mapsto\widetilde{\Phi}_1(\varphi;\epsilon)$ is the sum of the two functions

$$J_{11}^{q,\rho}(\varphi;\lambda) = \lambda\gamma_d\int\|Q\|^{2(\lambda-d)}_{q,\rho}\frac{\overline{\partial}\big(\sum_{j=1}^{d}\rho_j^2|Q_j|^2\big)}{\|Q\|^2_{q,\rho}}\wedge\Omega(s^{q,\rho,1};\{1,\ldots,d\})\wedge\bigwedge_{j=1}^{d}dQ_j\wedge\varphi$$

$$J_{12}^{q,\rho}(\varphi;\lambda) = \lambda\gamma_d\int\|Q\|^{2(\lambda-d)}_{q,\rho}\frac{\overline{\partial}\big(\sum_{j=d+1}^{d+m}\rho_j^2|Q_j|^2\big)}{\|Q\|^2_{q,\rho}}\wedge\Omega(s^{q,\rho,1};\{1,\ldots,d\})\wedge\bigwedge_{j=1}^{d}dQ_j\wedge\varphi$$

We consider now a point z_0 which is either an arbitrary point of $V(P)$, in the discrete case, or a regular point of one of the components $\widetilde{\Gamma}_\tau$, otherwise. In the first case, all the polynomials $Q_{d+1}=P_1,\ldots,Q_{d+m}=P_m$ vanish at the point z_0. In this case, it follows from the local Lojasiewicz inequality [JKS] (applied to $Q_1,\ldots,Q_d$, which also vanish at z_0), that the germs at z_0 of all the polynomials $Q_j^{D_1^d}$, $j=d+1,\ldots,d+m$, are in the integral closure of the ideal generated by the germs of $Q_1,\ldots,Q_d$. In the second case, since z_0 is a regular point both of $V(P)$ and of $V(Q_1,\ldots,Q_d)$ and these two algebraic varieties are purely $n-d$ dimensional, the first one being included into the second one, it follows that the two germs of variety they define at z_0 coincide. Therefore, the polynomials Q_j, $j=d+1,\ldots,d+m$, vanish on the germ of variety defined by $Q_1,\ldots,Q_d$ at z_0. As in the first case, it follows from local Lojasiewicz inequality [JKS] (applied to $Q_1,\ldots,Q_d$, which also vanish at z_0), that the germs at z_0 of all the polynomials $Q_j^{D_1^d}$, $j=d+1,\ldots,d+m$, are in the integral closure of the ideal generated by the germs of $Q_1,\ldots,Q_d$.

Let $\pi:\mathcal{X}_{z_0}\mapsto W(z_0)$ a resolution of singularities near z_0 for $\{P_1\cdots P_m=0\}$ such that in local coordinates on $\mathcal{X}_{z_0}$ (centered at a point y), one has, in the corresponding local chart U_y around y,

$$\pi^*Q_j(t)=u_j(t)t_1^{\alpha_{j,1}}\cdots t_n^{\alpha_{j,n}}=u_j(t)t^{\alpha_j},\ j=1,\ldots,d,$$

where the u_j are non vanishing holomorphic functions and at least one of the monomials $t^{\alpha_j}=\mu(t)$, $j=1,\ldots,d$, divides any t^{α_k}, $k=1,\ldots,d$. Since the $P_j^{D_1^d}$, $j=1,\ldots,m$ lie in the integral closure of the ideal generated by $Q_1,\ldots,Q_d$ near z_0, the monomial μ^d divides any $\pi^*(Q_l)=\pi^*P_{j-l}^{dD_1^d}$, $l=d+1,\ldots,d+m$. In the local coordinates t in the local chart

$$\pi^*\|Q\|^2_{q,\rho}=\Big(\pi^*\Big(\sum_{j=1}^{d}\rho_j^2|Q_j|^2\Big)\Big)(1+|\mu|^2\theta), \tag{3.4}$$

where θ is a positive real analytic function. If we express $J_{11}^{q,\rho}(\varphi;\lambda)$ as a sum of integrals on the local charts that cover $\pi^*(\mathrm{Supp}\,(\varphi))$ after rewriting it as

$$J_{11}^{q,\rho}(\varphi;\lambda) =$$
$$= \lambda\gamma_d \int \|Q\|_{q,\rho}^{2(\lambda-d)} \frac{\overline{\partial}\big(\sum_{j=1}^d \rho_j^2|Q_j|^2\big)}{\sum_{j=1}^d \rho_j^2|Q_j|^2} \wedge \Omega(s^{q,\rho,1};\{1,\dots,d\}) \wedge \bigwedge_{j=1}^d dQ_j \wedge \frac{\sum_{j=1}^d \rho_j^2|Q_j|^2}{\|Q\|_{q,\rho}^2}\varphi,$$

we see, using (3.4) in each local chart and the fact that the computations of $J_{11}^{q,\rho}(\varphi;0)$ involve only integration currents on the coordinate axis $\{t_j=0\}$ where t_j divides μ, that

$$J_{11}^{q,\rho}(\varphi;0) = \left[\lambda\gamma_d \int \|Q\|_{q,\rho}^{2(\lambda-d)} \frac{\overline{\partial}\big(\sum_{j=1}^d \rho_j^2|Q_j|^2\big)}{\sum_{j=1}^d \rho_j^2|Q_j|^2} \wedge \Omega(s^{q,\rho,1};\{1,\dots,d\}) \wedge \bigwedge_{j=1}^d dQ_j \wedge \varphi\right]_{\lambda=0}. \tag{3.5}$$

If we express the integrals in local coordinates, we can see (as it was extensively discussed in the proof of Lemma 1.2, and is based on the fact that one can essentially consider the ρ_j as constants when computing the values at zero of these meromorphic functions) that we also have

$$J_{11}^{q,\rho}(\varphi;0) = \left[\lambda\gamma_d \int \|Q\|_{q,\rho}^{2(\lambda-d)} \bigwedge_{j=1}^d \overline{\partial}(\rho_j|Q_j|^2) \wedge \bigwedge_{j=1}^d \partial(\log \rho_j|Q_j|^2) \wedge \varphi\right]_{\lambda=0}. \tag{3.6}$$

It follows from Proposition 8 in [BY2] (see also, for a more detailed proof, [PTY, Section 4]) that

$$J_{11}^{q,\rho}(\varphi;0) = \delta_{[(Q_1,\dots,Q_d)]}(\varphi),$$

where $\delta_{[(Q_1,\dots,Q_d)]}$ is the integration current (with multiplicities) on $\{Q_1=\ldots=Q_p=0\}$ near z_0. Since the local multiplicities at z_0 for the ideals $(Q_1,\dots,Q_d)$ and $(P_1,\dots,P_m)$ coincide, we have also

$$J_{11}^{q,\rho}(\varphi;0) = \delta_Z(\varphi)\,.$$

If we now express $J_{12}^{q,\rho}(\varphi;\lambda)$ or the Mellin transform of $\epsilon \to \widetilde{\Phi}_2(\varphi;\epsilon)$ in the desingularization coordinates, we see that these functions appear as combinations of terms of the form

$$\lambda \int_{U_y} \frac{|a\mu|^{2\lambda}}{\mu^d}\Big(\vartheta + \varpi \wedge \frac{\overline{\partial\mu}}{\overline{\mu}}\Big) \wedge (\pi^* P_j^N)\varphi, \tag{3.7}$$

where U_y is a local chart around y, μ the corresponding distinguished monomials, a a non vanishing function in U_y, ϑ and ϖ two smooth forms with respective types (d,d) and $(d,d-1)$, and $j \in \{1,\dots,m\}$. The choice of $N \geq dD_1^d$ implies that μ^d divides $\pi^* P_j^N$, so that the integrand in (3.7) has no holomorphic singularities. Therefore, the value at the origin of the meromorphic function defined by (3.7) is zero. So we have $J_{12}^{q,\rho}(\varphi;0) = \widetilde{\Phi}_2(\varphi;0) = 0$, which means that our current I_0 coincides with the integration current on Z (with multiplicities) near z_0. In the two cases (in the discrete case directly, and otherwise using the density in $V(P)$ of such points z_0), we conclude that $I_0 = \delta_Z$. ♢

Remark 3.1. It follows from formula (2.1) that $I_0(\varphi)$, which also equals the value at $\lambda = 0$ of the Mellin transform of $\epsilon \mapsto \widetilde{\Phi}(\varphi;\epsilon)$, is the value at $\lambda = 0$ of the meromorphic continuation of $\lambda \mapsto \frac{\lambda}{(2\pi i)^d}\int_{\mathbf{P}^n(\mathbf{C})} A_\lambda^{(d)} \wedge \varphi$, where the differential form $\lambda A_\lambda^{(d)}$ is the term involving λ as a factor in the decomposition

$$\begin{aligned} \left[\overline{\partial}(\|Q\|_{q,\rho}^{2\lambda}\log\|Q\|_{q,\rho}^2)\right]^d &= \overline{\partial}\Big[(\|Q\|_{q,\rho}^{2\lambda}\partial\log\|Q\|_{q,\rho}^2)\wedge\big(\overline{\partial}(\|Q\|_{q,\rho}^{2\lambda}\log\|Q|_{q,\rho}^2)\big)^{d-1}\Big] \\ &= \|Q\|_{q,\rho}^{2\lambda d}B^{(d)} + \lambda A_\lambda^{(d)}\,. \end{aligned} \tag{3.8}$$

Following the method developped in [BY2, section 4], one may now construct a Green current associated with a purely dimensional cycle Z in $\mathbf{P}^n(\mathbf{C})$, even if it is not defined as a complete intersection. The key point is that this current is computed in terms of generators of the ideal that define the cycle (with multiplicities). We proceed as follows. Let $\xi \mapsto L_\xi$ be the meromorphic map from $\mathbf{C}$ to $\mathcal{D}^{n,n}(\mathbf{P}^{2n+1}(\mathbf{C}))$ expressed in homogeneous coordinates (x,y) in $\mathbf{P}^{2n+1}(\mathbf{C})$ as

$$L_\xi := \frac{-1}{\xi}\left(\frac{||x-y||^2}{||x||^2+||y||^2}\right)^\xi\left(\sum_{k=0}^n\big(dd^c\log||x-y||^2\big)^k\wedge(dd^c\log(||x||^2+||y||^2))^{n-k}\right).$$

The value at $\xi = 0$ of this meromorphic map coincides with the Levine form ([GK],[Le]) for the subspace $x = y$ in $\mathbf{P}^{2n+1}(\mathbf{C})$; note that this subspace is defined as a complete intersection in $\mathbf{P}^{2n+1}(\mathbf{C})$. Let π the map from $(\mathbf{C}^{n+1})^* \times (\mathbf{C}^{n+1})^* \times (\mathbf{C}^2)^*$ to $\mathbf{P}^{2n+1}(\mathbf{C})$ obtained by taking quotients from the map

$$((\mathbf{C}^{n+1})^*)^2\times(\mathbf{C}^2)^*\mapsto(\mathbf{C}^{2(n+1)})^*:\ (x,y,(\beta_0,\beta_1))\mapsto(\beta_0x,\beta_1y)\,.$$

One can now define a meromorphic map $\xi \mapsto \Upsilon_\xi$ from $\mathbf{C}$ into the space of $(n-1,n-1)$ currents on $\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})$ as

$$\Upsilon_\xi(x,y) := \int_{\beta\in\mathbf{P}^1(\mathbf{C})}\pi^*(L_\xi)(x,y,\beta)\,.$$

For more details about this construction, we refer to [BY1, Section 4]. We now can state the following theorem.

Theorem 3.2. *Let Z be the effective algebraic cycle of pure dimension $n-d$ in $\mathbf{P}^n(\mathbf{C})$ which corresponds to the homogeneous ideal generated by the homogeneous polynomials $P_1,\dots,P_m$, with respective degrees $D_1 \geq \dots \geq D_m$. Let Λ be a generic linear form in $(x_0,\dots,x_n)$ and $\widetilde{Q}_1,\dots\widetilde{Q}_d$, d generic linear combinations of the polynomials $P_k\Lambda^{D_1-D_k}$, $k=1,\dots,m$. Let*

$$F = \sum_{j=1}^d\frac{|\widetilde{Q}_j|^2}{\|x\|^{2D_1}} + \sum_{k=1}^m\frac{|P_k|^{2(dD_1^d+1)}}{\|x\|^{2D_k(dD_1^d+1)}}$$

and Ω_1 and Ω_2 the singular (d,d) differential forms in $\mathbf{P}^n(\mathbf{C})$ defined by the formal identity

$$\frac{1}{(2\pi i)^d}\left[\overline{\partial}(F^\lambda \partial \log F)\right]^d = F^{d\lambda}[\Omega_1 + d\,\lambda\,\Omega_2]\,.$$

Then, the $(d-1,d-1)$ current-valued map $\lambda \mapsto \mathbf{G}_\lambda$ defined for any complex number λ with a large real part by

$$< \mathbf{G}_\lambda, \varphi >= \lambda^2 \int_{\mathbf{P}^n(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})} F^{\lambda^2}(y)\Omega_2(y)\wedge \Upsilon_\lambda(x,y)\wedge\varphi \tag{3.9}$$

can be analytically continued as a meromorphic function with a simple pole at $\lambda = 0$. The coefficient $\mathbf{G}_0$ of λ^0 in the Laurent development about the origin is a current which is smooth outside the support of Z and satisfies the Green equation

$$dd^c\mathbf{G}_0 + \delta_Z = (\deg Z)\omega^d\,. \tag{3.10}$$

Proof. It follows from Theorem 3.1 and Remark 3.1 that, for any $(n-d,n-d)$ test form in $\mathbf{P}^n(\mathbf{C})$, one has

$$\begin{aligned} < \delta_Z, \varphi > &= \left[\lambda \int_{\mathbf{P}^n(\mathbf{C})} F^\lambda(y)\Omega_2(y)\wedge\varphi(y)\right]_{\lambda=0} \\ &= \left[d\lambda \int_{\mathbf{P}^n(\mathbf{C})} F^{d\lambda}(y)\Omega_2(y)\wedge\varphi(y)\right]_{\lambda=0}. \end{aligned} \tag{3.11}$$

The proof of the proposition follows exactly the proof of Proposition 9 in [BY2]. The meromorphic map

$$\lambda \mapsto d\,\lambda\,F^{d\lambda}\Omega_2$$

plays the role of $\lambda \mapsto I_\lambda$. The identity (3.8)

$$\overline{\partial}\Big[(F^\lambda\partial\log F)\wedge\big(\overline{\partial}(F^\lambda\partial\log F)\big)^{d-1}\Big] = (2i\pi)^d F^{d\lambda}(\Omega_1 + \lambda\Omega_2)$$

can be written as

$$-\frac{1}{(2\pi i)^d}\overline{\partial}\Big[(F^\lambda\partial\log F)\wedge\big(\overline{\partial}(F^\lambda\partial\log F)\big)^{d-1}\Big] = -I_\lambda + \widetilde{I}_\lambda$$

and used exactly as the identity that defines $\widetilde{I}_\lambda$ in [BY2]. We will not repeat here the details of the proof. $\diamondsuit$

Let $\mathcal{Z}$ be an arithmetic cycle in $\operatorname{Proj}\mathbf{Z}[x_0,\dots,x_n]$, defined by m homogeneous polynomials $P_1,\dots,P_m$, with respective degrees $D_1 \geq \dots \geq D_m$. We assume that the algebraic cycle $Z = \mathcal{Z}(\mathbf{C})$ is purely dimensional, with codimension d. Then, one can compute the degree of Z as

$$\deg Z = \operatorname{Res}_{\lambda=0}\left[\int_{\mathbf{P}^n(\mathbf{C})} F^\lambda\Omega_2\wedge\omega^{n-d}\right],$$

where

$$F = \sum_{j=1}^{d} \frac{\big| \sum\limits_{k=1}^{m} \lambda_{jk} \Lambda^{D_1 - D_k} P_k \big|^2}{\|x\|^{2D_1}} + \sum_{k=1}^{m} \frac{|P_k|^{2(dD_1^d+1)}}{\|x\|^{2D_k(dD_1^d+1)}}$$

and Ω_2 is defined by the formal identity

$$\frac{1}{(2\pi i)^d} \left[\overline{\partial}(F^\lambda \partial \log F) \right]^d = F^{d\lambda} [\Omega_1 + d\,\lambda\,\Omega_2]\,,$$

the linear form Λ and the coefficients λ_{jk}, $j = 1, \dots, d$, $k = 1, \dots, m$, being generic.

If we assume that $\{x_0 = \cdots = x_{n-d} = P_1(x) = \cdots = P_m(x) = 0\}$ is the empty set in $\mathbf{P}^n(\mathbf{C})$, then the logarithmic size of $\mathcal{Z}$ (in the sense of [BGS]) is the sum of the "arithmetic" contribution

$$\sum_{\tau \text{ prime}} n_\tau \log \tau$$

(where $\sum\limits_{\tau \text{ prime}} n_\tau$ is the $n+1$ arithmetic cycle $\Pi \cdot \mathcal{Z}$, where $\Pi := \{x_0 = \cdots = x_{n-d} = 0\}$), and of an "analytic" contribution, which can be obtained as

$$\frac{\deg Z}{2} \sum_{k=d}^{n} \sum_{j=1}^{k} \frac{1}{j} - \frac{1}{2} \mathrm{Res}_{\lambda=0} \left[\lambda \int_{(x,y) \in \mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})} F^{\lambda^2}(y) \omega(x)^{n-d+1} \wedge \Omega_2(y) \wedge \Upsilon_\lambda(x,y) \right]$$
$$+ \frac{1}{2} \mathrm{Res}_{\lambda=0} \left[\lambda \int_{\Pi \times \mathbf{P}^n(\mathbf{C})} F^{\lambda^2}(y) \wedge \Omega_2(y) \wedge \Upsilon_\lambda(x'', y) \right].$$

Thus, we have a close expression for the degree and the analytic contribution in the expression of the size as residues at $\lambda = 0$ of zeta functions of λ that can be expressed in terms of the polynomials $P_1, \dots, P_m$ that define the ideal sheaf $I(Z)$. This result extends the result one could obtain before only for complex hypersurfaces (see the examples in [BY2] and [D]) and, more generally, for complete intersections see BY2. In fact, in the complete intersection case, computing a Green current is much simpler when the polynomials P_j have the same degree D. We let

$$\|P\|_\rho^2 = \sum_{k=1}^{m} \frac{|P_k(x)|^2}{\|x\|^{2D}}\,.$$

Proposition 3.3. *Let $P_1, \dots, P_d$, be d homogeneous polynomials in $n+1$ variables, with degree D, defining a complete intersection cycle Z in $\mathbf{P}^n(\mathbf{C})$. Then the $(d-1, d-1)$-current valued meromorphic map*

$$\lambda \mapsto \mathbf{G}_\lambda = \frac{-1}{\lambda} \|P\|_\rho^{2\lambda} \Big(\sum_{k=0}^{d-1} (dd^c \log \|P\|_\rho^2)^k \wedge (D\omega)^{d-1-k} \Big)$$

can be analytically continued as a meromorphic function in $\mathbf{C}$ *with a simple pole at* 0*. Moreover, the coefficient* $\mathbf{G}_0$ *of* λ^0 *in the Laurent development at the origin is a solution of the Green equation*

$$dd^c\mathbf{G}_0 + \delta_Z = D^d\omega^d\,.$$

Finally, the current $\mathbf{G}_0$ *is smooth at the origin.*

Remark 3.2. This proposition shows that the construction in Proposition 9 in [BY2] can be avoided in the complete intersection case. Nethertheless, this construction remains essential for the general case.

Proof. We compute, as in [BY2], formula (67),

$$dd^c\mathbf{G}_\lambda = \|P\|_\rho^{2\lambda}D^d\omega^d - \frac{i}{2\pi}\lambda\partial\log\|P\|_\rho^2\wedge\overline{\partial}\log\|P\|_\rho^2\wedge(dd^c\log\|P\|_\rho^2)^{d-1} + R_\lambda,$$

where

$$R_\lambda = -\frac{i}{2\pi}\lambda\|P\|_\rho^{2\lambda}\partial\log\|P\|_\rho^2\wedge\overline{\partial}\log\|P\|_\rho^2\wedge\Big(\sum_{k=0}^{d-2}(dd^c\log\|P\|_\rho^2)^k\wedge(D\omega)^{d-1-k}\Big)\,.$$

We have

$$\overline{\partial}\|P\|_\rho^2\partial\log\|P\|_\rho^2 = \|P\|_\rho^{2\lambda}\big(\lambda\overline{\partial}\log\|P\|_\rho^2\wedge\partial\log\|P\|_\rho^2 + \overline{\partial}\partial\log\|P\|_\rho^2\big)\,.$$

This implies, for any $k\geq 1$, that

$$\begin{aligned}&(\overline{\partial}\|P\|_\rho^2\partial\log\|P\|_\rho^2)^k =\\ &= \|P\|_\rho^{\lambda k}\Big((\overline{\partial}\partial\log\|P\|_\rho^2)^k + \lambda\overline{\partial}\log\|P\|_\rho^2\wedge\partial\log\|P\|_\rho^2\wedge(\overline{\partial}\partial\log\|P\|_\rho^2)^{k-1}\Big)\\ &= \|P\|_\rho^{\lambda k}B^{(k)} + \lambda A_\lambda^{(k)}\,.\end{aligned}$$

The function

$$\lambda\mapsto\int A_\lambda^{(k)}\wedge\varphi,\ \ \varphi\in\mathcal{D}^{n-k,n-k}(\mathbf{P}^n(\mathbf{C})),$$

is (up to a constant) the Mellin transform (with $k\lambda$ instead of λ) of the function

$$\epsilon\mapsto\frac{\gamma_k}{\epsilon^k}\int_{\|P\|_\rho^2=\epsilon}\Big[\sum_{\substack{i_1<\dots<i_k\\ 1\leq i_l\leq d}}\Omega(s;\{i_1,\dots,i_k\})\wedge\bigwedge_{l=1}^{k}dP_{i_l}\Big]\wedge\varphi$$

where $s_j = \|x\|^{-2D}\overline{P_j},\ j=1,\dots,d$ (see formula (2.3)). The value at 0 of this Mellin transform equals

$$\sum_{1\leq i_1<i_2<\dots<i_k\leq d}\text{Res}\begin{bmatrix} dP_{i_1}\wedge\cdots\wedge dP_{i_k}\wedge\varphi\\ P_{i_1},...,P_{i_k}\\ P_1,...,P_d\end{bmatrix}^{\underline{0},\rho}\,.$$

These sums of residue symbols are zero whenever $k < d$ (see Lemma 1.1). So, for any k between 0 and $d-2$, the current which is defined as the value at $\lambda = 0$ of

$$\lambda \mapsto \lambda \|P\|_\rho^{2\lambda} \partial \log \|P\|_\rho^2 \wedge \overline{\partial} \log \|P\|_\rho^2 \wedge (dd^c \log \|P\|_\rho^2)^k$$

is the zero current. Since, we have also (see [BY1, Proposition 8])

$$\Big[\frac{i}{2\pi} \lambda \partial \log \|P\|_\rho^2 \wedge \overline{\partial} \log \|P\|_\rho^2 \wedge (dd^c \log \|P\|_\rho^2)^{d-1}\Big]_{\lambda=0} = \delta_Z ,$$

we get at $\lambda = 0$ the relation

$$dd^c \mathbf{G}_0 + \delta_Z = D^d \omega^d .$$

It is clear that $\mathbf{G}_0$ is smooth outside the support of the cycle Z. $\diamondsuit$

Remark 3.3. When the P_j define a complete intersection, they have the same degree, their coefficients are in $\mathbf{Z}$, and they are such that $\Pi \cap V(P)$ is the empty set in $\mathbf{P}^n(\mathbf{C})$, where $\Pi = \{x_0 = \cdots = x_{n-d} = 0\}$, then, the analytic contribution to the arithmetic size of the cycle $\mathcal{Z}$ defined by the P_j in $\mathrm{Proj}\, \mathbf{Z}[x_0, \ldots, x_n]$ is

$$\frac{D^d}{2} \sum_{k=d}^{n} \sum_{j=1}^{k} \frac{1}{j} + \frac{1}{2} \mathrm{Res}_{\lambda=0} \frac{1}{\lambda^2} \Big[\int_{\mathbf{P}^n(\mathbf{C})} \|P\|_\rho^{2\lambda} \Big(\sum_{k=0}^{d-1} (dd^c \log \|P\|_\rho^2)^k \wedge (D\omega)^{n-1-k} \Big) \Big]$$
$$- \frac{1}{2} \mathrm{Res}_{\lambda=0} \frac{1}{\lambda^2} \Big[\int_{\Pi} \|P\|_\rho^{2\lambda} \Big(\sum_{k=0}^{d-1} (dd^c \log \|P\|_\rho^2)^k \wedge (D\omega)^{n-1-k} \Big) \Big].$$

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