BOUNDS FOR DEGREES AND SUMS OF DEGREES OF IRREDUCIBLE CHARACTERS OF SOME CLASSICAL GROUPS OVER FINITE FIELDS

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The goal of this note (which is incorporated in a forthcoming paper "The algebraic principle of the large sieve") is to bound from above in a suitable manner the degree of irreducible representations, and the sum of the degrees of irreducible representations, of a group G_ℓ, which in applications is either between $SL(r,\mathbf{F}_\ell)$ and $GL(r,\mathbf{F}_\ell)$, or between $Sp(2g,\mathbf{F}_\ell)$ and $CSp(2g,\mathbf{F}_\ell)$.

More generally, given a finite group G and $p\in[1,+\infty]$, we denote

$$A_p(G)=\Bigl(\sum_\rho \dim(\rho)^p\Bigr)^{1/p},\qquad \text{if } p\neq+\infty \text{ and}\qquad A_\infty(G)=\max\{\dim(\rho)\}$$

where ρ runs over irreducible linear representations of G (in characteristic zero). For example, we have $A_2(G)=\sqrt{|G|}$ for all G and if G is abelian, then $A_p(G)=|G|^{1/p}$ for all p. We are primarily interested in $A_1(G)$ and $A_\infty(G)$, but other cases may turn out to be useful.

We start with an easy monotonicity lemma.

Lemma 1. *Let G be a finite group and $J\subset G$ a subgroup, $p\in[1,+\infty]$. We have*

$$A_p(H)\leqslant A_p(G).$$

Proof. For any irreducible representation ρ of H, choose (arbitrarily) an irreducible representation $\pi(\rho)$ of G that occurs with positive multiplicity in the induced representation $\mathrm{Ind}_H^G\rho$.

Let π be a representation of G in the image of $\rho\mapsto\pi(\rho)$. For any ρ where $\pi(\rho)=\pi$, we have

$$\langle\rho,\mathrm{Res}_H^G\pi\rangle_H=\langle\mathrm{Ind}_H^G\rho,\pi\rangle_G>0,$$

by Frobenius reciprocity, (i.e., all ρ with $\pi(\rho)=\pi$ occur in the restriction of π to H. Hence

$$\sum_{\substack{\rho\\ \pi(\rho)=\pi}}\dim(\rho)^p\leqslant\Bigl(\sum_{\substack{\rho\\ \pi(\rho)=\pi}}\dim(\rho)\Bigr)^p\leqslant\dim(\pi)^p,$$

and summing over all possible $\pi(\rho)$ gives the inequality

$$A_p(H)^p\leqslant A_g(G)^p$$

by positivity. This settles the case $p\neq+\infty$, and the other case only requires noticing that $\dim(\rho)\leqslant\dim(\pi(\rho))\leqslant A_\infty(G)$. □

We come to the main result of this Appendix. The terminology is explained by examples after the proof.

Proposition 2. (1) *Let $\mathbf{G}/\mathbf{F}_q$ be a split connected reductive linear algebraic group of dimension d and rank r over a finite field, with connected center. Let W be its Weyl group and $G=\mathbf{G}(\mathbf{F}_q)$ the finite group of rational points of $\mathbf{G}$.*

For any subgroup $H\subset G$ and $p\in[1,+\infty]$, we have

$$A_p(H)\leqslant(q+1)^{(d-r)/2+r/p}\Bigl(1+\frac{2r|W|}{q-1}\Bigr),$$

with the convention $r/p=0$ if $p=+\infty$.

(2) *If $\mathbf{G}$ is a product of groups of type A or C, i.e., of linear and symplectic groups, then*

$$A_p(H)\leqslant(q+1)^{(d-r)/2+r/p}.$$

The proof is based on a simple interpolation argument from the extreme cases $p = 1$, $p = +\infty$. Indeed by Lemma 1 we can clearly assume $H = G$ and by writing the obvious inequality

$$A_p(G)^p = \sum_{\rho} \dim(\rho)^p \leqslant A_\infty(G)^{p-1} A_1(G),$$

we see that it suffices to prove the following:

Proposition 3 (J. Michel). *Let $\mathbf{G}/\mathbf{F}_q$ be a split connected reductive linear algebraic group of dimension d with connected center, and let $G = \mathbf{G}(\mathbf{F}_q)$ be the finite group of its rational points. Let r be the rank of $\mathbf{G}$. Then we have*

$$(1) \qquad A_\infty(G) \leqslant \frac{|G|_{p'}}{(q-1)^r} \leqslant (q+1)^{(d-r)/2}, \qquad \text{and} \qquad A_1(G) \leqslant (q+1)^{(d+r)/2}\Bigl(1 + \frac{2r|W|}{q-1}\Bigr),$$

where $n_{p'}$ denotes the prime-to-p part of a rational number n, p being the characteristic of $\mathbf{F}_q$. Moreover, if the principal series of G is not empty[1]*, there is equality*

$$A_\infty(G) = \frac{|G|_{p'}}{(q-1)^r}$$

and $\dim \rho = A_\infty(G)$ *if and only if ρ is in the principal series.*

Finally if $\mathbf{G}$ is a product of groups of type A or C, then the factor $(1 + 2r|W|/(q-1))$ may be removed in the bound for $A_1(G)$.

It seems quite possible that the factor $(1 + 2r|W|/(q-1))$ can always be removed, but we haven't been able to figure this out using Deligne-Lusztig characters, and in fact for groups of type A or C, we simply quote *exact formulas* for $A_1(G)$ due to Gow, Klyachko and Vinroot, which are proved in completely different ways. The "right" upper bound for the case of groups of type A may be recovered using the structure of unipotent representations of such groups.

Note that the extra factor is not likely to be a problem in many applications where $q \to +\infty$, but it may be questionable for uniformity with respect to the rank.

Proof. This is based on properties of the Deligne-Lusztig generalized characters. We will mostly refer to [DM], [Ca] and [L] for all facts which are needed (using notation from [DM], except for writing simply G for what is denoted G^F there). We identify irreducible representations of G (up to isomorphism) with their characters seen as complex-valued functions on G.

First, for a connected reductive group $\mathbf{G}/\mathbf{F}_q$ over a finite field, Deligne and Lusztig have constructed (see e.g. [DM, 11.14]) a family $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ of generalized representations of $G = \mathbf{G}(\mathbf{F}_q)$ (i.e., linear combinations with integer coefficients of "genuine" representations of G), parameterized by pairs $(\mathbf{T}, \theta)$ consisting of a maximal rational torus (i.e., defined over $\mathbf{F}_q$) $\mathbf{T} \subset \mathbf{G}$ and a (one-dimensional) character θ of the finite abelian group $T = \mathbf{T}(\mathbf{F}_q)$. The $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ are not all irreducible, but any irreducible character occurs (with positive or negative multiplicity) in the decomposition of at least one such character. Moreover, $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ only depends (up to isomorphism) on the G-conjugacy class of the pair $(\mathbf{T}, \theta)$.

We quote here a useful classical fact: for any $\mathbf{T}$ we have

$$(2) \qquad (q-1)^r \leqslant |T| \leqslant (q+1)^r$$

(see e.g. [DM, 13.7 (ii)]), and moreover $|T| = (q-1)^r$ if and only if $\mathbf{T}$ is a split torus (i.e., $\mathbf{T} \simeq \mathbf{G}_m^r$ over $\mathbf{F}_q$). Indeed, we have

$$|T| = |\det(q^n - w \mid Y_0)|$$

where $w \in W$ is such that $\mathbf{T}$ is obtained from a split torus $\mathbf{T}_0$ by "twisting with w" (see e.g. [Ca, Prop. 3.3.5]), and $Y_0 \simeq \mathbf{Z}^r$ is the group of cocharacters of $\mathbf{T}$. If $\lambda_1, \ldots, \lambda_i$ are the eigenvalues of w acting on Y_0, which are roots of unity, then we have

$$|T| = \prod_{i=1}^{r} (q - \lambda_i),$$

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[1] In particular if q is large enough given $\mathbf{G}$.

and so $|T| = (q-1)^r$ if and only if each λ_i is equal to 1, if and only if w acts trivially on Y_0, if and only if $w = 1$ and $\mathbf{T}$ is split.

As in [DM, 12.12], we denote by $\rho \mapsto p(\rho)$ the orthogonal projection of the space $\mathcal{C}(G)$ of real-valued conjugacy-invariant functions on G to the subspace generated by Deligne-Lusztig characters, where $\mathcal{C}(G)$ is given the standard scalar product

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)g(x),$$

and for a representation ρ, we of course denote $p(\rho) = p(\operatorname{Tr}\rho)$ the projection of its character.

For any representation ρ, we have $\dim(\rho) = \dim(p(\rho))$, where $\dim(f)$, for an arbitrary function $f \in \mathcal{C}(G)$ is obtained by linearity from the degree of characters. Indeed, for any f standard character theory shows that

$$\dim(f) = \langle f, \mathrm{reg}_G\rangle$$

where reg_G is the regular representation of G. From [DM, 12.14], the regular representation is in the subspace spanned by the Deligne-Lusztig characters, so by definition of an orthogonal projector we have

$$\dim(\rho) = \langle \rho, \mathrm{reg}_G\rangle = \langle p(\rho), \mathrm{reg}_G\rangle = \dim(p(\rho)).$$

Now because the characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for distinct conjugacy classes of $(\mathbf{T}, \theta)$ are orthogonal (see e.g. [DM, 11.15]), we can write

$$p(\rho) = \sum_{(\mathbf{T},\theta)} \beta(\mathbf{T}, \theta) R_{\mathbf{T}}^{\mathbf{G}}(\theta)$$

where

$$\beta(\mathbf{T}, \theta) = \frac{\langle \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle},$$

and so

$$\dim(p(\rho)) = \sum_{(\mathbf{T},\theta)} \beta(\mathbf{T}, \theta) \dim(R_{\mathbf{T}}^{\mathbf{G}}(\theta)).$$

By [DM, 12.9] we have

$$\dim(R_{\mathbf{T}}^{\mathbf{G}}(\theta)) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}|G|_{p'}|T|^{-1}, \tag{3}$$

where $\varepsilon_{\mathbf{G}} = (-1)^r$ and $\varepsilon_{\mathbf{T}} = (-1)^{r(\mathbf{T})}$, $r(\mathbf{T})$ being the $\mathbf{F}_q$-rank of $\mathbf{T}$ (see [DM, p. 66] for the definition). This yields the formula

$$\dim(p(\rho)) = |G|_{p'} \sum_{(\mathbf{T},\theta)} \frac{1}{|T|} \frac{\langle \rho, \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}. \tag{4}$$

Now we use the fact that pairs $(\mathbf{T}, \theta)$ are partitioned in *geometric conjugacy classes*, defined as follows: two pairs $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ are geometrically conjugate if and only if the generalized characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ have a common irreducible component (see e.g. [DM, 13.2]). In particular, for a given ρ, if $\langle \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle$ is non-zero for some $(\mathbf{T}, \theta)$, then by definition only pairs $(\mathbf{T}', \theta')$ geometrically conjugate to $(\mathbf{T}, \theta)$ may satisfy $\langle \rho, R_{\mathbf{T}'}^{\mathbf{G}}(\theta)\rangle \neq 0$. So we have

$$\dim(p(\rho)) = |G|_{p'} \sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{|T|} \frac{\langle \rho, \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle},$$

for some geometric conjugacy class κ, depending on ρ. By Cauchy-Schwarz, we have

$$\dim(p(\rho)) \leqslant |G|_{p'} \Big(\sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{|T|^2} \frac{1}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}\Big)^{1/2} \Big(\sum_{(\mathbf{T},\theta)\in\kappa} \frac{|\langle \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle|^2}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}\Big)^{1/2}. \tag{5}$$

The second term on the right is simply $\langle p(\rho), p(\rho)\rangle \leqslant \langle \rho, \rho\rangle = 1$. As for the first term we have

$$\sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{|T|^2} \frac{1}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle} \leqslant \frac{1}{(q-1)^{2r}} \sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}$$

by (2). Now it is known that for each class κ, the assumption that $\mathbf{G}$ has connected center implies that the generalized characters

$$\chi(\kappa) = \sum_{(\mathbf{T},\theta)\in\kappa} \frac{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta)}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}$$

is in fact an irreducible character of G (such characters are called *regular* characters). This implies that

$$\sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle} = \langle \chi(\kappa), \chi(\kappa)\rangle = 1,$$

and so we have

$$\dim p(\rho) \leqslant \frac{|G|_{p'}}{(q-1)^r}. \tag{6}$$

Now observe that we will have equality in this argument if ρ is itself of the form $\pm R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, and if $|T| = (q-1)^r$. Those conditions hold for representations of the principal series, i.e., characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for an $\mathbf{F}_q$-split torus $\mathbf{T}$ and a character θ "in general position" (see e.g. [Ca, Cor. 7.3.5]). Such characters are also, more elementarily, induced characters $\mathrm{Ind}_B^G(\theta)$, where $B = \mathbf{B}(\mathbf{F}_q)$ is a Borel subgroup containing T, for some Borel subgroup $\mathbf{B}$ defined over $\mathbf{F}_q$ containing $\mathbf{T}$ (which exist for a split torus $\mathbf{T}$); there θ is extended to B by setting $\theta(u) = 1$ for unipotent elements $u \in B$. For this, see e.g. [L, Prop.2.6].

Conversely, let ρ be such that

$$\dim \rho = \frac{|G|_{p'}}{(q-1)^r}$$

and let κ be the associated geometric conjugacy class. From the above, for any $(\mathbf{T}, \theta)$ in κ, we have $|T| = (q-1)^r$, i.e., $\mathbf{T}$ is $\mathbf{F}_q$-split. Now it follows from Lemma 4 (probably well-known) that this implies that the geometric conjugacy class κ contains a single pair $(\mathbf{T}, \theta)$, and then $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is an irreducible representation (e.g. from the definition of $\chi(\kappa)$), so must be equal to ρ.

We now come to $A_1(G)$. To deal with the fact that in (4), $|T|$ depends on $(\mathbf{T}, \theta) \in \kappa$, we write

$$\dim(p(\rho)) = \frac{|G|_{p'}}{(q-1)^r} \sum_{\kappa} \langle \rho, \chi(\kappa)\rangle + |G|_{p'} \sum_{(\mathbf{T},\theta)} \Big(\frac{1}{|T|} - \frac{1}{(q-1)^r}\Big) \frac{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}\langle \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}$$

(since by (2), the dependency is rather weak).

Consider the first term's contribution. Since $\chi(\kappa)$ is an irreducible character, the sum

$$\sum_{\rho}\sum_{\kappa} \langle \rho, \chi(\kappa)\rangle$$

is simply the number of geometric conjugacy classes. This is given by $q^{r'}|Z|$ by [DM, 14.42] or [Ca, Th. 4.4.6 (ii)], where r' is the semisimple rank of $\mathbf{G}$ and $Z = Z(\mathbf{G})(\mathbf{F}_q)$ is the group of rational points of the center of $\mathbf{G}$. For this quantity, note that the center of $\mathbf{G}$ being connected implies that $Z(\mathbf{G})$ is the radical of $\mathbf{G}$ (see e.g. [Sp, Pr. 7.3.1]) so $Z(\mathbf{G})$ is a torus and $r = r' + \dim Z(\mathbf{G})$. So using again the bounds (2) for the cardinality of the group of rational points of a torus, we obtain

$$|Z|q^{r'} \leqslant (q+1)^r. \tag{7}$$

To estimate the sum of the contributions in the second term, say $\sum t(\rho)$, we write

$$\sum_\rho t(\rho) = |G|_{p'} \sum_{(\mathbf{T},\theta)} \Big(\frac{1}{|T|} - \frac{1}{(q-1)^r}\Big) \frac{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} \langle \sum_\rho \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle},$$

and we bound

$$\Big|\langle \sum_\rho \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle\Big| \leqslant \langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle$$

for any $(\mathbf{T},\theta)$, since we can write

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \sum_\rho a(\rho)\rho \quad \text{with } a(\rho) \in \mathbf{Z},$$

and therefore

$$\Big|\langle \sum_\rho \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle\Big| = \Big|\sum_\rho a(\rho)\Big| \leqslant \sum_\rho |a(\rho)|^2 = \langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle.$$

Thus

$$\sum_\rho t(\rho) \leqslant \frac{|G|_{p'}}{(q-1)^r} \frac{2r}{q-1} |\{(\mathbf{T},\theta)\}|.$$

There are at most $|W|$ different choices of $\mathbf{T}$ up to G-conjugacy, and for each there are at most $|T| \leqslant (q+1)^r$ different characters, and so we have

$$\sum_\rho t(\rho) \leqslant \frac{|G|_{p'}}{(q-1)^r} \frac{2r|W|}{q-1} (q+1)^r$$

and

$$\sum_\rho \dim \rho \leqslant (q+1)^r \frac{|G|_{p'}}{(q-1)^r} \Big(1 + \frac{2r|W|}{q-1}\Big). \tag{8}$$

To conclude, we use the classical formula

$$|G| = q^N \prod_{1 \leqslant i \leqslant r} (q^{d_i} - 1),$$

where N is the number of positive roots of $\mathbf{G}$, and the d_i are the degrees of reflections of the Weyl group (this is because G is split; see e.g. [Ca, 2.4.1 (iv); 2.9, p. 75]). So

$$|G|_{p'} = \prod_{1 \leqslant i \leqslant r} (q^{d_i} - 1,$$

and

$$\frac{|G|_{p'}}{(q-1)^r} = \prod_{1 \leqslant i \leqslant r} \frac{q^{d_i} - 1}{q-1} \leqslant \prod_{1 \leqslant i \leqslant r} (q+1)^{d_i - 1} = (q+1)^{\sum (d_i - 1)} = (q+1)^{(d-r)/2}, \tag{9}$$

since $\sum (d_i - 1) = N$ and $N = (d-r)/2$ (see e.g.[Ca, 2.4.1], [Sp, 8.1.3]).

Inserting this in (6) we derive the first inequality in (1), and with (8), we get

$$A_1(G) \leqslant (q+1)^{(d+r)/2} \Big(1 + \frac{2r|W|}{q-1}\Big),$$

which is the second part of (1).

Now we explain why the extra factor involving the Weyl group can be removed for products of groups of type A and C. Clearly it suffices to work with $\mathbf{G} = GL(n)$ and $\mathbf{G} = CSp(2g)$.

For $\mathbf{G} = GL(n)$, with $d = n^2$ and $r = n$, Gow [Go] and Klyachko [K] have proved independently that $A_1(G)$ is equal to the number of symmetric matrices in G. The bound

$$A_1(G) \leqslant (q+1)^{(n^2+n)/2}$$

follows immediately.

For $\mathbf{G} = CSp(2g)$, with $d = 2g^2 + g + 1$ and $r = g + 1$, the exact analog of Gow's theorem is due to Vinroot [V]. Again, Vinroot's result implies $A_1(G) \leqslant (q+1)^{(d+r)/2}$ in this case (see [V, Cor 6.1], and use the formulas for the order of unitary and linear groups to check the final bound). $\square$

Here is the lemma used in the determination of $A_\infty(G)$ when there is a character in general position of a split torus:

Lemma 4. *Let $\mathbf{G}/\mathbf{F}_q$ be a split connected reductive linear algebraic group of dimension d and let $G = \mathbf{G}(\mathbf{F}_q)$ be the finite group of its rational points. Let $\mathbf{T}$ be a split torus in $\mathbf{G}$, θ a character of T. If $\mathbf{T}'$ is also a split torus for any pair $(\mathbf{T}', \theta')$ geometrically conjugate to $(\mathbf{T}, \theta)$, then the geometric conjugacy class of $(\mathbf{T}, \theta)$ is the singleton $\{(\mathbf{T}, \theta)\}$.*

Proof. Consider $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. If it is irreducible, then clearly we are done. Otherwise, by the scalar product formula for Deligne-Lusztig characters, there exists $w \in W$, $w \neq 1$, such that ${}^w\theta = \theta$ (see e.g. [DM, Cor. 11.15]). Let $\mathbf{T}'$ be a torus obtained from $\mathbf{T}$ by "twisting by w", i.e., $\mathbf{T}' = g\mathbf{T}g^{-1}$ where $g \in \mathbf{G}$ is such that $g^{-1}\operatorname{Fr}(g) = w$ (see e.g. [Ca, 3.3]). Let $Y = \operatorname{Hom}(\mathbf{G}_m, \mathbf{T}) \simeq \mathbf{Z}^r$ (resp Y') be the abelian group of cocharacters of $\mathbf{T}$ (resp.$\mathbf{T}$;); the conjugation isomorphism $\mathbf{T} \rightarrow \mathbf{T}'$ gives rise to a conjugation isomorphism $Y \rightarrow Y'$ (loc. cit.). Moreover, there is an action of the Frobenius Fr on Y and a canonical isomorphism $T \simeq Y/(\operatorname{Fr}-1)Y$ (see e.g. [DM, Prop. 13.7]), hence canonical isomorphisms of the character groups $\hat{T}$ and $\hat{T}'$ as subgroups of the characters groups of Y and Y':

$$\hat{T} \simeq \{\chi : Y \rightarrow \mathbf{C}^\times \mid (\operatorname{Fr}-1)Y \subset \ker\chi\}, \quad \hat{T}' \simeq \{\chi : Y' \rightarrow \mathbf{C}^\times \mid (\operatorname{Fr}-1)Y' \subset \ker\chi\}.$$

Unravelling the definitions, a simple calculation shows that the condition ${}^w\theta = \theta$ is precisely what is needed to prove that the character χ of Y associated to θ, when "transported" to a character χ' of Y' by the conjugation isomorphism, still satisfies $\ker\chi' \supset (\operatorname{Fr}-1)Y'$ (see in particular [Ca, Prop. 3.3.4]), so is associated with a character $\theta' \in \hat{T}'$.

It is then clear (see the characterization of geometric conjugacy in [DM, Prop. 13.8]) that $(\mathbf{T}, \theta)$ is geometrically conjugate to $(\mathbf{T}', \theta')$, and since $w \neq 1$, the torus $\mathbf{T}'$ is not split, this means that the geometric conjugacy class of $(\mathbf{T}, \theta)$ contains two elements at least. $\square$

Example 5. (1) Let ℓ be prime, $r \geqslant 1$ and let $\mathbf{G} = GL(r)/\mathbf{F}_\ell$. Then $G = GL(r, \mathbf{F}_\ell)$, $\mathbf{G}$ is a split connected reductive of rank r and dimension r^2, with connected center of dimension 1. So from Lemma 1 and Proposition 2, we get

$$A_p(H) \leqslant (\ell+1)^{r(r-1)/2 + r/p}$$

for $p \in [1, +\infty]$ for any subgroup H of G, and in particular

$$A_\infty(H) \leqslant (\ell+1)^{r(r-1)/2} \quad \text{and} \quad A_1(H) \leqslant (\ell+1)^{r(r+1)/2}$$

It would be interesting to know if there are other values of p besides $p = 1$, 2 and $+\infty$ (the latter when q is large enough) for which $A_p(GL(n, \mathbf{F}_q))$ can be computed exactly.

(2) Let $\ell \neq 2$ be prime, $g \geqslant 1$ and let $\mathbf{G} = CSp(2g)/\mathbf{F}_\ell$. Then $G = CSp(2g, \mathbf{F}_\ell)$ and G is a split connected reductive group of rank $g+1$ and dimension $2g^2 + g + 1$, with connected center. So from Lemma 1 and Proposition 2, we get

$$A_p(H) \leqslant (\ell+1)^{g^2 + (g+1)/p}$$

for $p \in [1, +\infty]$ for any subgroup H of G, and in particular

$$A_\infty(H) \leqslant (\ell+1)^{g^2} \quad \text{and} \quad A_1(H) \leqslant (\ell+1)^{g^2+g+1}.$$

Remark 6. Here is a mnemotechnical way to remember the bounds for $A_\infty(G)$ in (1)[2]: among the representatons of G, we have the principal series $R(\theta)$, a family parameterized by the

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[2] Which explains why it seemed to the author to be a reasonable statement to look for...

characters of a maximal split torus, of which there are about q^r, and those share a common maximal dimension Δ. Hence

$$q^r\Delta^2 = \sum_{\theta} \dim(R(\theta))^2 \simeq |G| \sim q^d,$$

so Δ is of order $q^{(d-r)/2}$. In other words: in the formula $\sum \dim(\rho)^2 = |G|$, the principal series contributes a positive proportion.

The bound for $A_1(G)$ is also intuitive : there are roughly q^r conjugacy classes, and for a "positive proportion" of them, the degree of the representation is of the maximal size given by $A_\infty(G)$.

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