Approximation of the MHD equations in heterogeneous domains using Lagrange finite elements.

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Dynamo effect

Dynamo effect: "generation of a non vanishing magnetic field by a moving ferromagnetic fluid".

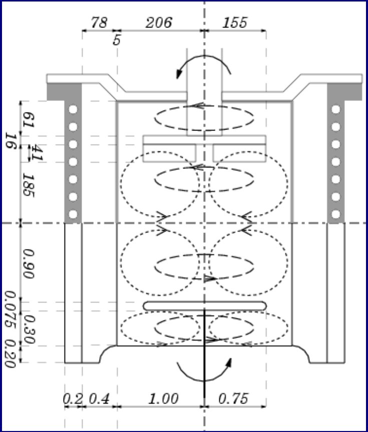
- Moving incompressible fluid $\rightsquigarrow$ Navier-Stokes equations:

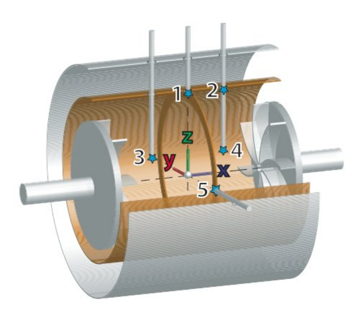
$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} - R_e^{-1} \Delta \mathbf{u} + \nabla p = (\nabla \times \mathbf{H}) \times \mu \mathbf{H}$$

- Ferromagnetic fluid $\rightsquigarrow$ Maxwell equations:

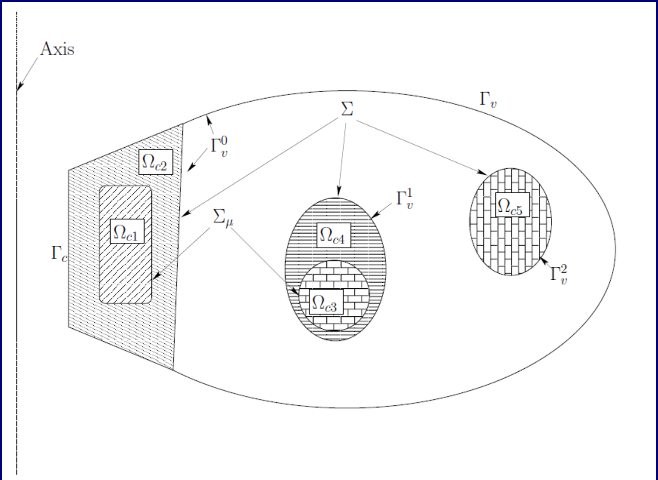
$$\mu \partial_t \mathbf{H} + \nabla \times \mathbf{E} = 0$$
$$\nabla \times \mathbf{H} = R_m \sigma \left(\mathbf{E} + \mathbf{u} \times \mu \mathbf{H}\right) + \mathbf{j}^s$$

Van Kármán Sodium Experiment





Generic axisymmetric domain



$$
\begin{cases}
\mu\partial_t \mathbf{H} = -\nabla\times\mathbf{E} & \text{in } \Omega \\
\nabla\times\mathbf{H} = R_m\sigma\left(\mathbf{E}+\mathbf{u}\times\mu\mathbf{H}\right)+\mathbf{j}^s & \text{in } \Omega_c \\
\nabla\times\mathbf{H} = 0 & \text{in } \Omega_v \\
\nabla\cdot\mathbf{E} = 0 & \text{in } \Omega_v \\
\mathbf{H}^c\times\mathbf{n}^c+\mathbf{H}^v\times\mathbf{n}^v = 0 & \text{on } \Sigma \\
\mathbf{E}^c\times\mathbf{n}^c+\mathbf{E}^v\times\mathbf{n}^v = 0 & \text{on } \Sigma \\
[\![\mathbf{H}]\!]\times\mathbf{n} = 0 & \text{on } \Sigma_\mu \\
[\![\mathbf{E}]\!]\times\mathbf{n} = 0 & \text{on } \Sigma_\mu \\
\mathbf{E}\times\mathbf{n} = \mathbf{a} & \text{on } \Gamma
\end{cases}
$$

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\left\{
\begin{array}{rcll}
\mu\partial_t \mathbf{H} & = & -\nabla\times\mathbf{E} & \text{in } \Omega \\
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\nabla\times\mathbf{H} & = & 0 & \text{in } \Omega_v \\
\nabla\cdot\mathbf{E} & = & 0 & \text{in } \Omega_v \\
\mathbf{H}^c\times\mathbf{n}^c+\mathbf{H}^v\times\mathbf{n}^v & = & 0 & \text{on } \Sigma \\
\mathbf{E}^c\times\mathbf{n}^c+\mathbf{E}^v\times\mathbf{n}^v & = & 0 & \text{on } \Sigma \\
[\![\mathbf{H}]\!]\times\mathbf{n} & = & 0 & \text{on } \Sigma_\mu \\
[\![\mathbf{E}]\!]\times\mathbf{n} & = & 0 & \text{on } \Sigma_\mu \\
\mathbf{E}\times\mathbf{n} & = & \mathbf{a} & \text{on } \Gamma
\end{array}
\right.
$$

$\mathbf{H}$: magnetic field
$\mathbf{E}$: electric field

$\mathbf{j}^s$: current
$\mathbf{u}$: velocity
+ boundary conditions
+ initial data

R_m : magnetic Reynolds number
σ : Conductivity
μ : Permeability

Reducing the number of unknowns

- Eliminate $\mathbf{E}^c$ using Ampère's equation.

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Functional framework

$$\mathbf{L} = \left\{(\mathbf{b},\psi) \in \mathbf{L}^2(\Omega_c) \times H^1_{\int=0}(\Omega_v)\right\}$$

$$\mathbf{X} = \left\{(\mathbf{b},\psi) \in \mathbf{H}_{\text{curl}}(\Omega_c) \times H^1_{\int=0};\, (\mathbf{b}\times\mathbf{n}^c + \nabla\psi\times\mathbf{n}^v)_{|\Sigma} = 0\right\}$$

$$(\mu^v \partial_t \nabla\phi, \nabla\psi)_{\Omega_v} = -(\nabla\times\mathbf{E}^v, \nabla\psi)_{\Omega_v}$$

$$
\begin{aligned}
(\mu^v \partial_t \nabla\phi, \nabla\psi)_{\Omega_v} &= -(\nabla\times\mathbf{E}^v, \nabla\psi)_{\Omega_v} \\
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\end{aligned}
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\left(\mu^c \partial_t \mathbf{H}^c, \mathbf{b}\right)_{\Omega_v} &= -\left(\mathbf{E}^c, \mathbf{b}\right)_{\Omega_c} + \left(\mathbf{E}^c\times\mathbf{n}^c, \mathbf{b}\right)_{\Gamma_c} \\
&- \left(\left\{\mathbf{E}^c\right\}, [\![\mathbf{b}]\!]\times\mathbf{n}\right)_{\Sigma_\mu} - \left(\mathbf{E}^c, \mathbf{b}\times\mathbf{n}^c\right)_{\Gamma_c}
\end{aligned}
$$

and then get rid of $\mathbf{E}^c$

IP method

$$
\begin{aligned}
& \left(\mu^c \partial_t \mathbf{H}^c, \mathbf{b}\right)_{\Omega_c} + \left(\mu^v \partial_t \nabla\phi, \nabla\psi\right)_{\Omega_v} \\
+ \quad & \Big((R_m\sigma)^{-1} \nabla\times\mathbf{H}^c - \mathbf{u}\times\mu^c\mathbf{H}^c, \nabla\times\mathbf{b} \Big)_{\Omega_c} \\
+ \quad & \Big((R_m\sigma)^{-1} \nabla\times\mathbf{H}^c - \mathbf{u}\times\mu^c\mathbf{H}^c, \mathbf{b}\times\mathbf{n}^c + \nabla\psi\times\mathbf{n}^v \Big)_{\Sigma} \\
+ \quad & \Big(\Big\{ (R_m\sigma)^{-1} \nabla\times\mathbf{H}^c - \mathbf{u}\times\mu^c\mathbf{H}^c \Big\}, [\![\mathbf{b}]\!]\times\mathbf{n} \Big)_{\Sigma_\mu} \\
= \quad & \ell\left(\mathbf{b}, \psi\right)
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+ \quad & h^{-1}\left(\mathbf{H}^c\times\mathbf{n}^c + \nabla\phi\times\mathbf{n}^v, \mathbf{b}\times\mathbf{n}^c + \nabla\psi\times\mathbf{n}^v\right)_{\Sigma} \\
+ \quad & h^{-1}\left([\![\mathbf{H}^c]\!]\times\mathbf{n}, [\![\mathbf{b}]\!]\times\mathbf{n}\right)_{\Sigma_\mu} \\
= \quad & \ell(\mathbf{b}, \psi)
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SFEMaNS

Spectral/Finite Element for Maxwell and Navier-Stokes equations: F90 code developed since 2002 by J.-L. Guermond, C. Nore, J. Léorat, R. Laguerre, A. Ribeiro and F.L.

- takes advantage of the cylindrical symmetry,
- Fourier decomposition in the azimuthal direction,
- Lagrange Finite Element solver in meridian plane,
- $\rightsquigarrow$ smaller systems,
- divergence of $\mu\mathbf{H}$ used to be stabilized in $\mathbf{L}^2$.

Aim: improve it to correctly solve problems involving eigenvalues, piecewise smooth permeability and/or geometrical singularities.

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Eigenvalue problem

For non-smooth μ (e.g. piecewise constant), find $\lambda, \mathbf{E}$ s.t.

$$\begin{aligned} \nabla\times\nabla\times\mathbf{E} &= \lambda\mu\mathbf{E} && \text{in } \Omega,\\ \nabla\cdot(\mu\mathbf{E}) &= 0 && \text{in } \Omega,\\ \mathbf{E}\times\mathbf{n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

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Requirements

- use Lagrange finite element

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Requirements

- use Lagrange finite element
- use low order polynomials

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Requirements

- use Lagrange finite element
- use low order polynomials
- use as less as possible information about Ω

Boundary value problem

First consider, for $\mathbf{E} \in \mathbf{H}$ the following

$$\left\{ \begin{array}{l} \text{find } \mathbf{F} \in \mathbf{X} \text{ such that} \\ \nabla\times\nabla\times\mathbf{F} = \mu\mathbf{E} \end{array} \right.$$

with :

$$\begin{aligned} \mathbf{H} &:= \left\{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla\cdot(\mu\mathbf{F}) = 0\right\} \\ \mathbf{H}_{0,\mathrm{curl}}(\Omega) &:= \left\{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla\times\mathbf{F} \in \mathbf{L}^2(\Omega) \text{ and } \mathbf{F}\times\mathbf{n}_{|\partial\Omega} = 0\right\} \\ \mathbf{X} &:= \mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H} \end{aligned}$$

Variational problem

Problem

$$\begin{cases} \text{find } \mathbf{F} \in \mathbf{X} \text{ such that } \forall \mathbf{B} \in \mathbf{X} \\ (\nabla \times \mathbf{F}, \nabla \times \mathbf{B}) = (\mu \mathbf{E}, \mathbf{B}) \end{cases}$$

We will write $\mathbf{F} = A\mathbf{E}$.

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We will write $\mathbf{F} = A\mathbf{E}$.

- the bilinear form is coercive on $\mathbf{X}$ $\rightsquigarrow$ A is well-defined.
- we have an eigenvalue problem for A.
- A can be defined on $L^2(\Omega)$.
- we have to deal with the divergence-free constraint.

Requirements on the numerical scheme

Spectral Convergence Result (Osborn 1975)

Assume

- (Pointwise Convergence) For all $\mathbf{E} \in \mathbf{L}^2$, $\lim_{h\to 0} \|(A_h - A)\mathbf{E}\|_{\mathbf{L}^2} = 0$;
- (Collective Compactness) For all U bounded set of $\mathbf{L}^2$, $\{A_h\mathbf{E}; \mathbf{E} \in U,\ 0 < h < 1\}$ is relatively compact in $\mathbf{L}^2$.

Then A_h is spectrally convergent to A.

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Then A_h is spectrally convergent to A.

- A is compact (Bonito and Guermond '10, Bonito, Guermond and L. '10)
- For $\mu = 1$ and $\mathbf{E} \in \mathbf{H}(\mathrm{div} = 0)$, we have $A\mathbf{E} \in \mathbf{H}^{1/2}$ and $\nabla\times A\mathbf{E} \in \mathbf{H}^{1/2}$.

Lagrange FE schemes for constant μ

Non-smooth domains (Costabel et al., '91)

If Ω is non-smooth and non-convex, the space $\mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}^1$ is a closed proper subset of $\mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}_{\mathrm{div}}(\Omega)$.

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Rehabilitation of Continuous Nodal Elements

- Dauge and Costabel ('02) , Bramble, Kolev and Pasciak ('05): control of the divergence in an intermediate space between $\mathbf{L}^2$ and $\mathbf{H}^{-1}$
- add $(w_\gamma \nabla \cdot A\mathbf{E}, w_\gamma \nabla \cdot \mathbf{B})$ to the bilinear form (Buffa, Ciarlet and Jamelot, '10).
- $w_\gamma \sim d^\gamma$, with d =distance to the singular edges/vertices.
- γ depends on the regularity of the domain.

Numerical scheme (I)

$$\left\{\begin{array}{l} \text{Let } 1/2 < \alpha < 1 \text{ and find } A_h\mathbf{E} \in \mathbf{X}_h \text{ such that } \forall \mathbf{B}_h \in \mathbf{X}_h \\ (\nabla\times A_h\mathbf{E}, \nabla\times\mathbf{B}_h) + \langle \nabla\cdot(\mu A_h\mathbf{E}), \nabla\cdot(\mu\mathbf{B}_h)\rangle_{-\alpha} = (\mu\mathbf{E}, \mathbf{B}) \end{array}\right.$$

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Case $\mu = 1$ (Bonito and Guermond, '09)

- Pointwise convergence for $\mathbf{E} \in \mathbf{L}^2$ and $\alpha \in (1/2, 1]$.
- Collective compactness for $\alpha < 1$.
- $\rightsquigarrow$ spectrally correct approximation for $1/2 < \alpha < 1$.

But $\langle \cdot, \cdot \rangle_{-\alpha}$ is not implementable.

Numerical scheme (II)

New scheme: Find $A_h\mathbf{E} \in \mathbf{X}_h$,

$$(\nabla\times A_h\mathbf{E}, \nabla\times\mathbf{B}_h) + \langle\nabla\cdot(\mu A_h\mathbf{E}), \nabla\cdot(\mu\mathbf{B}_h)\rangle_{-\alpha} = (\mu\mathbf{E}, \mathbf{B}_h), \qquad \forall\mathbf{B}_h \in \mathbf{X}_h$$

Numerical scheme (II)

New scheme: Find $A_h\mathbf{E} \in \mathbf{X}_h$,

$$(\nabla\times A_h\mathbf{E}, \nabla\times\mathbf{B}_h) + h^{2(\alpha-1)}\langle\nabla\cdot(\mu A_h\mathbf{E}), \nabla\cdot(\mu\mathbf{B}_h)\rangle_{-1} = (\mu\mathbf{E}, \mathbf{B}_h), \qquad \forall\mathbf{B}_h \in \mathbf{X}_h$$

From $\mathbf{H}^{-\alpha}$ to $\mathbf{H}^{-1}$, inverse estimate

$$\|\nabla\cdot(\mu\mathbf{B}_h)\|^2_{\mathbf{H}^{-\alpha}} \lesssim h^{2(\alpha-1)}\|\nabla\cdot(\mu\mathbf{B}_h)\|^2_{\mathbf{H}^{-1}}.$$

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$$(\mu A_h\mathbf{E}, \nabla q_h) - h^{2(\alpha-1)}(\nabla p_h, \nabla q_h) = 0, \qquad \forall q_h$$

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From $\mathbf{H}^{-1}$ to a mixed formulation

$$h^{2(\alpha-1)}\langle\nabla\cdot(\mu A_h\mathbf{E}), \nabla\cdot(\mu\mathbf{B}_h)\rangle_{\mathbf{H}^{-1}} = -(\nabla\cdot(\mu\mathbf{B}_h), \underbrace{h^{2(\alpha-1)}(-\Delta)^{-1}\nabla\cdot(\mu A_h\mathbf{E})}_{:=p_h})$$

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Inf-Sup stable scheme: $h^{2\alpha}\|\nabla\cdot(\mu\mathbf{B}_h)\|^2_{\mathbf{L}^2}$.

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Inf-Sup stable scheme: $h^{2\alpha}\|\nabla\cdot(\mu\mathbf{B}_h)\|^2_{\mathbf{L}^2}$.

Lemma: Discrete Control of $\nabla\cdot(\mu\mathbf{B}_h)$ in $H^{-\alpha}$ (Bonito and Guermond, '09)

$$\|\nabla\cdot(\mu\mathbf{B}_h)\|_{H^{-\alpha}} \le \sup_{q_h\in\mathbb{Q}_h} \frac{(\nabla\cdot(\mu\mathbf{B}_h), q_h)}{h^{1-\alpha}\|\nabla q_h\|_{\mathbf{L}^2}} + h^{\alpha}\|\nabla\cdot(\mu\mathbf{B}_h)\|_{\mathbf{L}^2}.$$

- Bonito and Guermond ('09): if $\mu = 1$, we have a spectrally correct approximation, provided $\dfrac{k}{2k-1} < \alpha < 1$.
- The only requirement on $\mathbb{Q}_h$ is that it is a subspace of H_0^1.
- Bonito, Guermond and L. ('10?): if α is sufficiently close to 1, we have a spectrally correct approximation.
- If $\nabla\cdot(\mu\mathbf{E}) \neq 0$, the order of convergence decreases when α increases.

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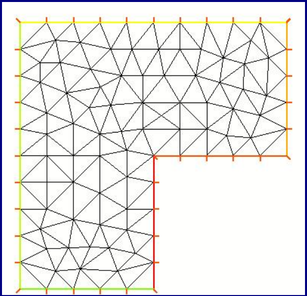
Boundary value problem, $\alpha = 0.75$



$$\mathbf{E} = \nabla\varphi$$

$$\varphi = r^{2/3}\sin\left(\frac{2}{3}\theta\right)$$

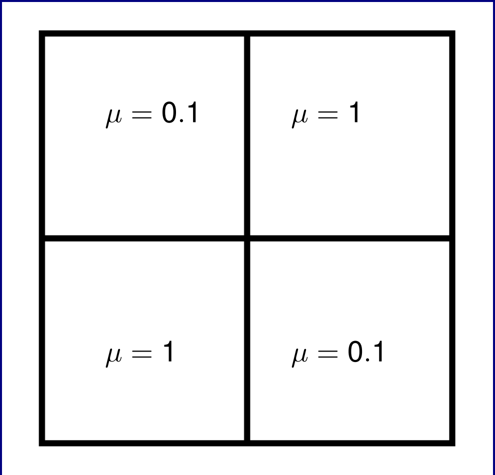
$\mathbb{P}_1$		
1/h	rel.err	coc
10	$2.390\,10^{-1}$	N/A
20	$1.843\,10^{-1}$	0.38
40	$1.405\,10^{-1}$	0.39
80	$1.031\,10^{-1}$	0.45
160	$7.544\,10^{-2}$	0.45

$\mathbb{P}_2$		
1/h	rel. err.	coc
10	$1.290\,10^{-1}$	N/A
20	$8.178\,10^{-2}$	0.66
40	$5.978\,10^{-2}$	0.45
80	$3.759\,10^{-2}$	0.67
160	$2.232\,10^{-2}$	0.75

Eigenvalue Problem, $\alpha = 0.7$

	$\lambda_1 \simeq 1.476$			$\lambda_2 \simeq 3.534$		
1/h	val.	rel. err.	coc	val.	rel. err.	coc
10	1.707	$1.452\,10^{-1}$	N/A	3.537	$8.266\,10^{-4}$	N/A
20	1.623	$9.522\,10^{-2}$	0.61	3.535	$2.380\,10^{-4}$	1.8
40	1.586	$7.240\,10^{-2}$	0.4	3.534	$6.640\,10^{-5}$	1.8
80	1.545	$4.614\,10^{-2}$	0.65	3.534	$1.726\,10^{-5}$	1.9
	$\lambda_3 = \pi^2 \simeq 9.870$			$\lambda_5 \simeq 11.389$		
1/h	val.	rel. err.	coc	val.	rel. err.	coc
10	7.828	$2.307\,10^{-1}$	N/A	7.903	$3.614\,10^{-1}$	N/A
20	9.870	$3.799\,10^{-7}$	19.21	11.39	$2.374\,10^{-5}$	13.89
40	9.870	$3.856\,10^{-8}$	3.3	11.39	$7.786\,10^{-6}$	1.61
80	9.870	$3.444\,10^{-8}$	0.16	11.39	$2.168\,10^{-6}$	1.85

Benchmark Problem



Eigenvalue Problem (II), $\alpha = 0.95$

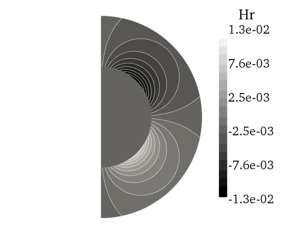
	$\lambda_1 \simeq 4.534$			$\lambda_2 \simeq 6.250$		
1/h	val.	rel. err.	coc	val.	rel. err.	coc
5	4.538	$8.358\,10^{-4}$	N/A	7.047	$1.274\,10^{-1}$	N/A
10	4.534	$9.592\,10^{-5}$	3.12	7.038	$1.261\,10^{-1}$	0.01
20	4.534	$3.992\,10^{-5}$	1.26	6.764	$8.218\,10^{-2}$	0.62
40	4.534	$1.606\,10^{-5}$	1.31	6.506	$4.096\,10^{-2}$	1.00
	$\lambda_3 \simeq 7.037$			$\lambda_4 \simeq 22.342$		
1/h	val.	rel. err.	coc	val.	rel. err.	coc
5	9.076	$2.897\,10^{-1}$	N/A	22.51	$7.489\,10^{-3}$	N/A
10	7.404	$5.220\,10^{-2}$	2.47	22.36	$9.487\,10^{-4}$	3.05
20	7.037	$2.274\,10^{-5}$	11.1	22.34	$9.935\,10^{-5}$	3.26
40	7.037	$2.597\,10^{-6}$	3.13	22.34	$9.718\,10^{-6}$	3.35

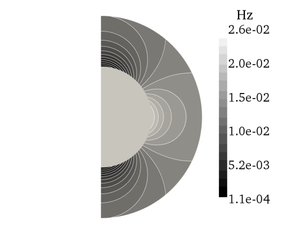
Conclusions and Open Problems

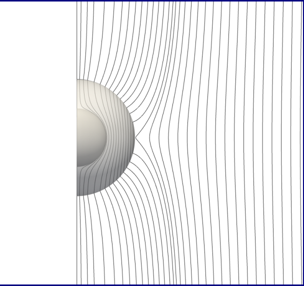
- $\mathbf{H}^1$ conforming elements produce a convergent spectral approximation of the Maxwell system provided that the divergence of the electric field is controlled in $\mathbf{H}^{-\alpha}$, $1/2 < \alpha < 1$.
- α close to 1 is required;
- small α are better for the compactness (spurious eigenvalues);
- The finite element solver needs improvement.

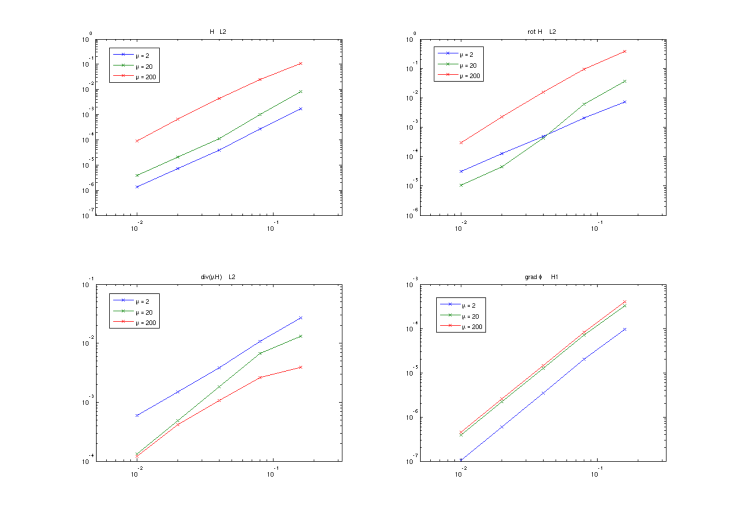
OUTLINE

Durand spheres

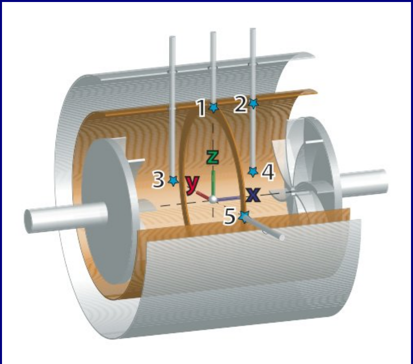






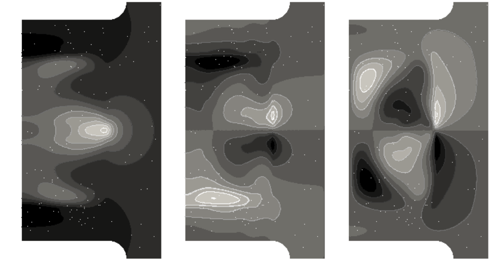

H L2
rot H L2
div(μH) L2
grad ϕ H1
μ = 2
μ = 20
μ = 200

VKS setting

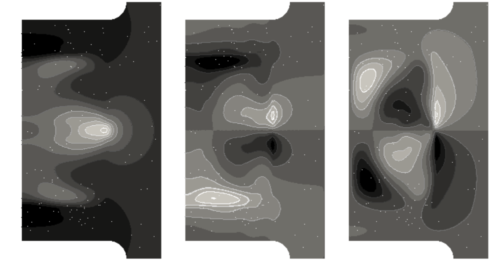


- copper envelope,
- $R_m \leqslant 50$ for the real experiment,
- impellers made of stainless steel $\rightsquigarrow$ no dynamo,
- impellers made of soft iron $\rightsquigarrow$ dynamo.

Stainless steel impellers

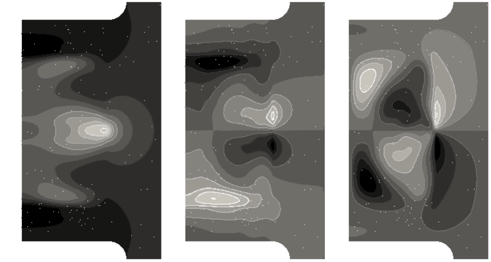


Stainless steel impellers



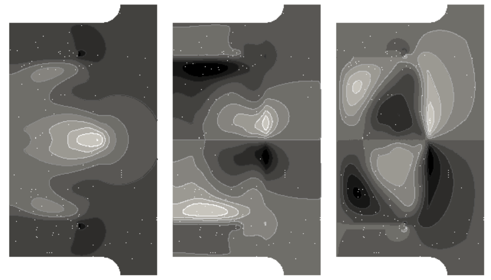
- Critical Reynolds number : $R_{mc} > 70$,

Stainless steel impellers

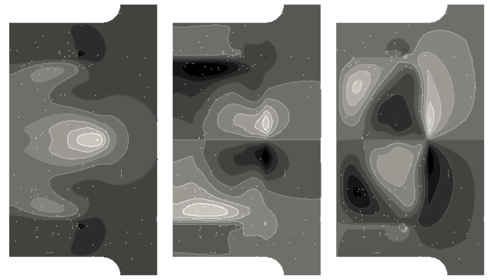


- Critical Reynolds number : $R_{mc} > 70$,
- Effect of the "lid-flow".

Soft iron impellers

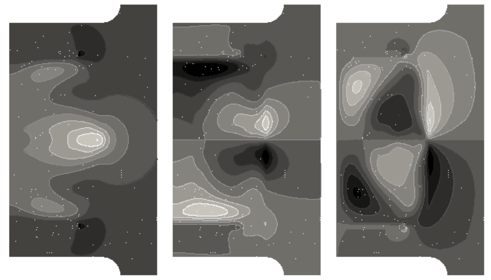


Soft iron impellers

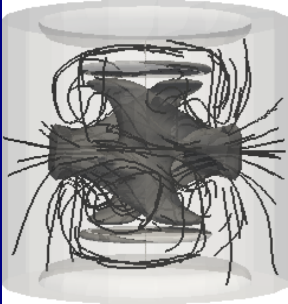


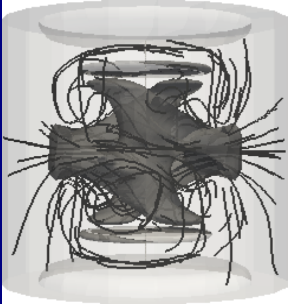
- Critical Reynolds number : $R_{mc} \sim 60$ close to the real setting,

Soft iron impellers



- Critical Reynolds number : $R_{mc} \sim 60$ close to the real setting,
- (almost) no effect of the “lid-flow”.





THANK YOU