# Theoretical and numerical analysis of MHD equations: application to dynamo effect

F. Luddens, J.-L. Guermond, J. Léorat, C. Nore, A. Ribeiro
Laboratoire d'Informatique pour la Mécanique et les Sciences de l'Ingénieur, Université Paris Sud XI

## Motivation

We have been developing numerical techniques to simulate fluid dynamos since 2003. We then have a numerical code SFEMaNS which is capable of integrating the incompressible Navier-Stokes and Maxwell equations in axisymmetric domains made of heterogeneous conducting regions and insulating regions. The approximation is done by using a mixed Fourier/Lagrange finite element technique. Finite element is used in the meridian plane, and Fourier approximation is done in the azimuthal direction. Continuity conditions across interfaces are enforced using an interior penalty method [3]. Moreover, parallelization is done with respect to the Fourier modes, and then in the meridian planes. The main challenge for the use of Lagrange finite element is the possibly discontinuous magnetic permeability, which has led us to set up a non-standard technique [2]. Then we have been able to investigate the effect of soft iron impellers in the VKS 2 experiment, which seems to play a key role in the dynamo effect. In particular, we have been able to study the effect of a magnetic permeability jump on the dynamo threshold in a VKS-like set up.

## 2D test case

### Strong formulation

We want to solve the following problem in the L-shape domain: for $\mathbf{f}$ a divergence free field, find $\mathbf{H}$ such that

$$\nabla\times\nabla\times\mathbf{H} = \mathbf{f}$$
$$\nabla\cdot\mathbf{H} = 0$$
$$\mathbf{H}\times\mathbf{n}_{|\Gamma} = 0$$

For any real value $C$, it is equivalent to

$$\nabla\times\nabla\times\mathbf{H} + \nabla p = \mathbf{f}$$
$$\nabla\cdot\mathbf{H} - C\Delta p = 0$$
$$\mathbf{H}\times\mathbf{n}_{|\Gamma} = 0 \quad , p_{|\Gamma} = 0$$

### Variationnal formulation

We use $C$ depending on the typical mesh size $h$. Given $\alpha > 0$, we want to solve

$$\int_\Omega \mathbf{f}\cdot\mathbf{F} = \int_\Omega \nabla\times\mathbf{H}\cdot\nabla\times\mathbf{F} + \int_\Omega \nabla p\cdot\mathbf{F} - \int_\Omega \mathbf{H}\cdot\nabla q + h^{2(1-\alpha)}\int_\Omega \nabla p\cdot\nabla q + \underbrace{h^{2\alpha}\int_\Omega \nabla\cdot\mathbf{H}\,\nabla\cdot\mathbf{F}}_{\text{stabilization term}}$$

In this formulation,

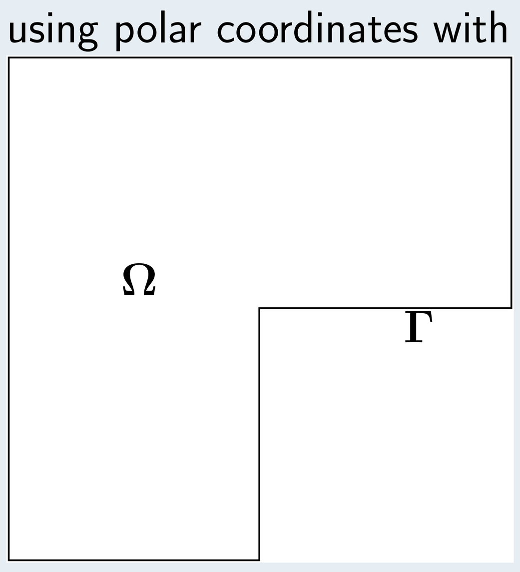
$$\mathbf{H},\mathbf{F} \in \{\mathbf{b}\in\mathbf{L}^2 \;/\; \nabla\times\mathbf{b}\in\mathbf{L}^2,\ \nabla\cdot\mathbf{b}\in L^2,\ \mathbf{b}\times\mathbf{n}_{|\Gamma}=0\}$$
$$p,q \in H^1_0(\Omega)$$

### Singular solution

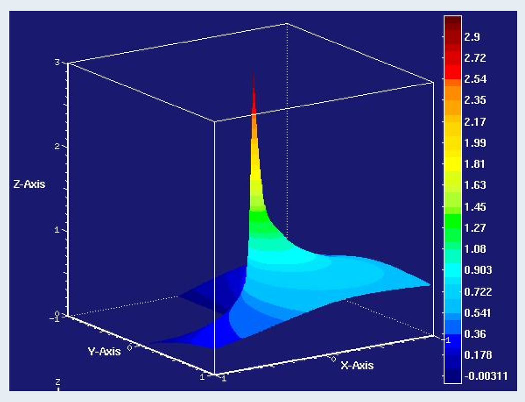
We use $\mathbf{f} = 0$ and a non homogeneous boundary condition on $\mathbf{H}\times\mathbf{n}$, so that the solution is

$$\mathbf{H} = \nabla\phi \text{ with } \phi(\rho,\theta) = \rho^{2/3}\sin\left(\frac{2}{3}\theta\right)$$

using polar coordinates with $\theta\in[0,2\pi[$.

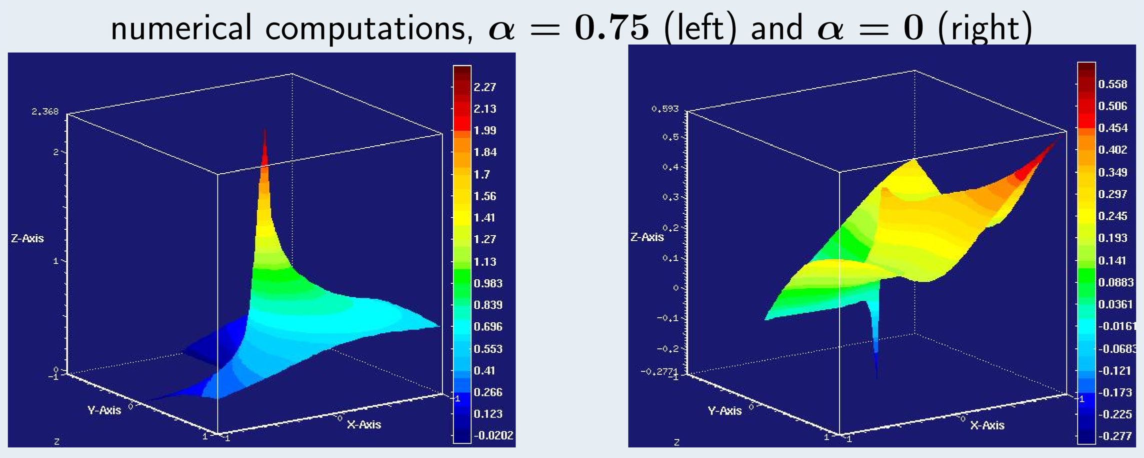



L-shape domain



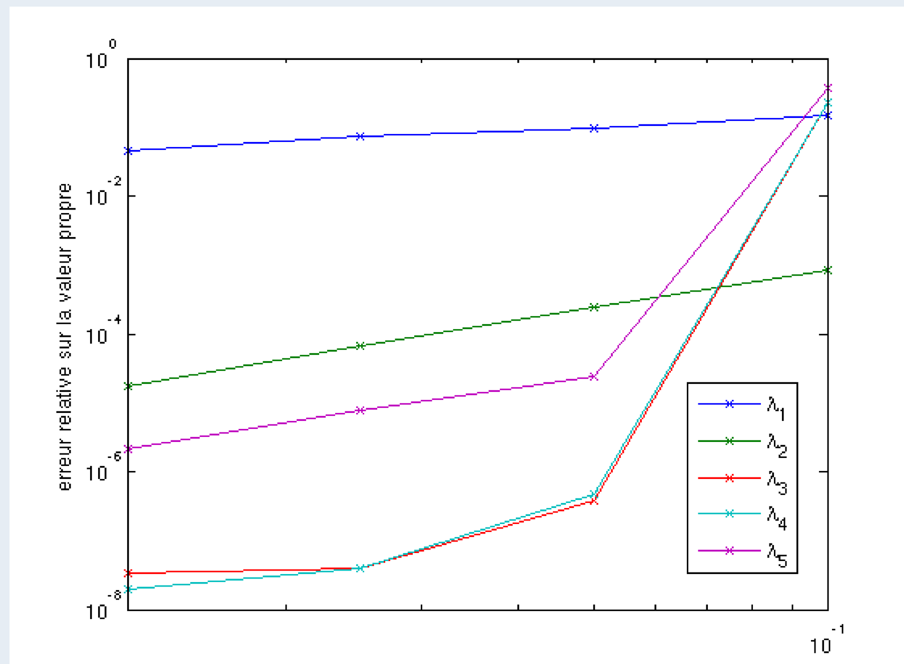
real solution

numerical computations, $\alpha = 0.75$ (left) and $\alpha = 0$ (right)



- The case $\alpha = 0$ corresponds to standard techniques. It has been proven that it may fail to converge to the right solution, owing to a lack of regularity of the solution [1]. Using $\alpha$ close to 1 gives good approximation results.
- In the case $\alpha\in[\frac{1}{2},1]$, we have convergence of the numerical solution to the right solution.
- In the case $\alpha\in[\frac{1}{2},1)$, we also have convergence of the numerical eigenvalues and eigenvectors, i.e. we have approximations of the pairs $(\mathbf{H},\lambda)$, such that

$$\nabla\times\nabla\times\mathbf{H} = \lambda\mathbf{H},\ \nabla\cdot\mathbf{H} = 0,\ \mathbf{H}\times\mathbf{n}_{|\Gamma} = 0$$



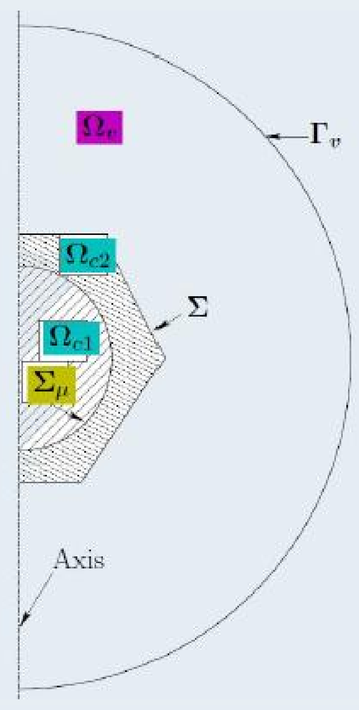

Approximation of the first five eigenvalues for 2D Maxwell problem in the L-shape domain. The worst convergence rate is $2/3$ for the singular eigenvector, and we recover $2$ for smoother eigenvectors.

## Application to MHD problems

### Strong formulation, case $\alpha = 1$

We use a formulation $\mathbf{H}-\phi$ where $\mathbf{H}$ (resp. $\phi$) denotes the magnetic field (resp. potential) in the conducting (resp. insulating) region. Relying on the 2D case, we introduce $p^c$ (resp. $p^v$) to stabilize the divergence in the conducting (resp. insulating) part.

$$\begin{cases} \mu\partial_t\mathbf{H} = -\nabla\times\mathbf{E} - \mu\nabla p^c & \text{in } \Omega_c \\ \nabla\cdot(\mu\mathbf{H}) - \nabla\cdot(\mu\nabla p^c) = 0 & \text{in } \Omega_c \\ p^c = 0 & \text{on } \partial\Omega_c \\ \nabla\times\mathbf{H} = R_m\sigma(\mathbf{E} + \mathbf{u}\times\mu\mathbf{H}) + \mathbf{j}^s & \text{in } \Omega_c \\ \nabla\cdot\mathbf{E} = 0 & \text{in } \Omega_c \\ \mu\partial_t\Delta\phi = -\mu\Delta p^v & \text{in } \Omega_v \\ \Delta\phi - \Delta p^v = 0 & \text{in } \Omega_v \\ \nabla p^v\cdot\mathbf{n} = 0 & \text{on } \partial\Omega_v \\ [\![\mathbf{H}\times\mathbf{n}]\!] = 0 & \text{in } \Sigma_\mu \end{cases}$$




### VKS setting

We test a kinematic dynamo on a VKS-like setting. The domain is divided into three parts :

- For the bulk flow region, we use an axisymmetric time-averaged flow provided by a water experiment [4].
- We model the disks and the blades with a "disk region" for which we prescribed the velocity field to be equal to the one of the bulk region at the interface between the two regions (NB: this is not a solid rotation model).
- For the lid flow, we combine two types of flow: $\mathbf{u}^\theta$ which we define as a linear interpolation between $0$ and the disk flow velocity field (so that both boundary and interface conditions are satisfied) and $\mathbf{u}^{pol}$ which is an analytical poloidal recirculation flow. The normalization is such that the maximum of $\mathbf{u}^{pol}$ is $10\%$ of the maximum of $\mathbf{u}^\theta$.

### Discrete formulation

- Using integration by parts and continuity conditions we can get rid of $\mathbf{E}$ and $p^v$,
- We use interior penalty method to enforce the continuity conditions accross the interfaces $\Sigma$ and $\Sigma_\mu$, by adding to the bilinear form the following

$$h^{-1}\int_{\Sigma_\mu}[\![\mathbf{H}\times\mathbf{n}]\!]\cdot[\![\mathbf{b}\times\mathbf{n}]\!] + h^{-1}\int_\Sigma(\mathbf{H}-\nabla\phi)\times\mathbf{n}\cdot(\mathbf{b}-\nabla\psi)\times\mathbf{n},$$

with $h$ the typical mesh size, $\mathbf{b}$ and $\psi$ two test-functions.

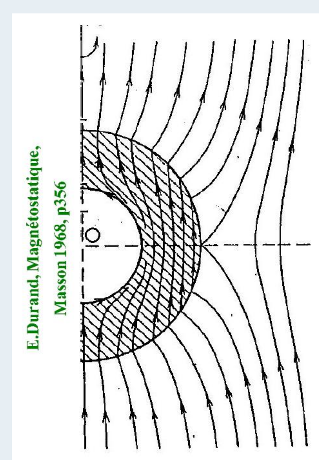
- We add a stabilization term for the divergence. In this case, it would be

$$h^2\int_{\Omega_c}\nabla\cdot(\mu\mathbf{H})\,\nabla\cdot(\mu\mathbf{b}).$$

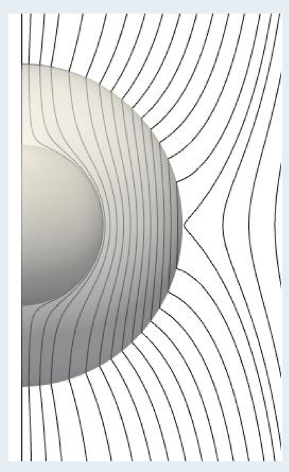
- Once again, we can introduce a parameter $\alpha$. In practice, we take $\alpha = 0,75$.

### Induction in a composite sphere

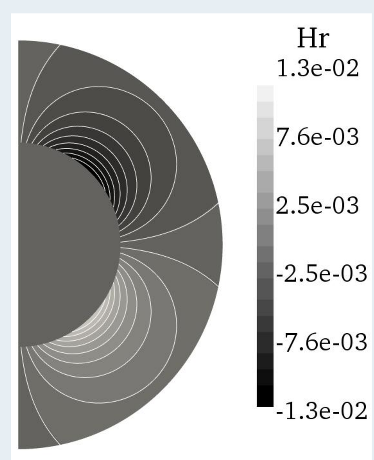
We validate the new formulation on the so-called Durand sphere. The setting is the following: the domain is made of two concentric conducting spheres with different permeabilities ($\mu_1$ for the inner sphere, $\mu_2$ for the rest of the conductor) in the vacuum. The magnetic field at infinity is a vertical and uniform field. The computations have been done with $\mu_1 = 1$ and $\mu_2 = 200$



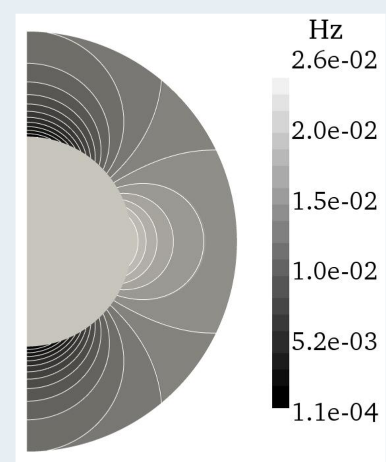
Analytical magnetic lines ($\mu_1 = 1,\ \mu_2\to\infty$)



Numerical magnetic lines in $\mathbf{B} = \mu\mathbf{H}$

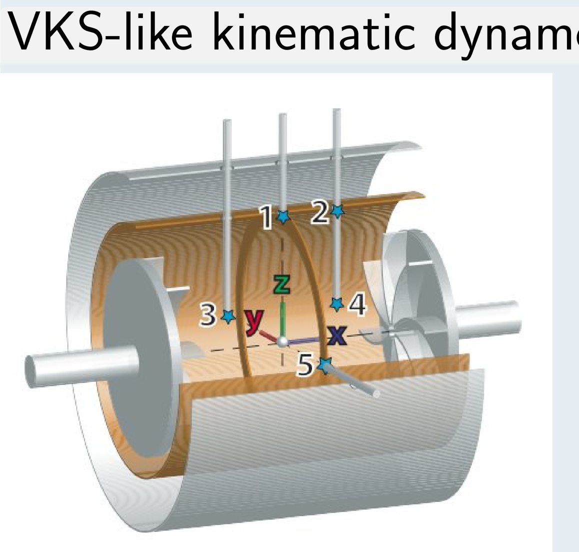



Radial component of $\mathbf{H}$

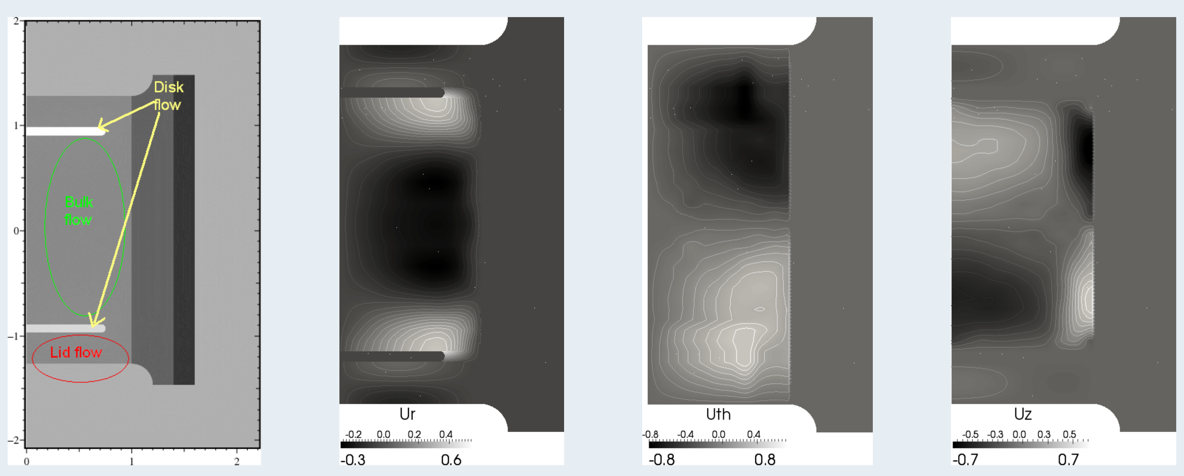



Vertical component of $\mathbf{H}$

### VKS-like kinematic dynamo



Schema of the setting




Left: Numerical domain. Right: Prescribed axisymmetric velocity field

| lid flow | material | $R_{mc}$ |
|---|---|---|
| $\mathbf{u}^\theta$ | steel | 82 |
| $\mathbf{u}^\theta + \mathbf{u}^{pol}$ | steel | 75 |
| $\mathbf{u}^\theta$ | soft iron | 66 |
| $\mathbf{u}^\theta + \mathbf{u}^{pol}$ | soft iron | 64 |

Computed critical Reynolds number with different lid flows

Magnetic lines and iso-value of the magnetic energy density (25% of the maximum magnetic energy): soft iron (left) and steel (right)

### Conclusions and Prospects

- For the VKS setting with soft iron, the lid flow seems to have no influence on the critical magnetic Reynolds number.
- Using soft iron instead of steel does affect the magnetic lines near the disks.
- The code is now capable of dealing properly with permeability jumps and geometric singularities.
- Another line of research is the simulation of the precession dynamo. The code has thus been modified to take into account the Coriolis term: computations are in progress.

### References

[1] M. Costabel (1991) A Remark on the Regularity of Solutions of Maxwell's Equations on Lipschitz Domains, Math. Methods Appl. Sci., 12(4):365-368

[2] J.-L. Guermond *et al.* (2011), Effects of discontinuous magnetic permeability on magnetodynamic problems, J. Comput. Phys. **230** 6299-6319

[3] R. Laguerre *et al.* (2009) Nonlinear magnetohydrodynamics in axisymmetric heterogeneous domains using a Fourier/finite element technique and an interior penalty method, J. Comput. Phys. **228** 2739-2757

[4] F. Ravelet *et al.* (2005) Towards an experimental von Kármán dynamo: numerical studies for an optimized design, Phys. Fluids 17:117104


contact: luddens@limsi.fr