Enablers for high order level set methods in fluid mechanics

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Motivations

An efficient way of representing and tracking the evolution of a interface is the level set method introduced by Osher and Sethian [1]. It consists of representing the interface as the zero level set of a higher dimensional function. This approach eases the treatment of topological changes and gives straightforward formula to compute geometric properties of the interface. Based on the work of Cisternino and Weynans [2] on elliptic problems with discontinuities across an interface, we want to design a sharp cartesian method for the Navier-Stokes equations. It requires accurate tracking of an interface and accurate computation of its geometric properties, even for long times computations. We develop a method in a finite differences framework on cartesian grids. The method involves reinitialization steps, i.e. computations of the signed distance function, which are shown to be necessary. However, their number has to be limited. We first design a fast but accurate method for the reinitialization step, and use it only when distortion of the iso contours is too important. The method compares well with usual level set strategies [3].

Context of the study

Level set ϕ, interface $\Gamma = \{\phi = 0\}$, curvature κ, typical mesh size h, total number of dof N.

Passively advected interface, i.e.

$$\partial_t \phi + \mathbf{U} \cdot \nabla \phi = 0.$$

$\mathbf{U}$ is the fluid velocity, divergence-free, available in the entire computational domain.
From time to time, reinitialization of the level set, through the eikonal equation

$$|\nabla \phi| = 1$$
$$\phi = 0 \text{ on } \Gamma$$

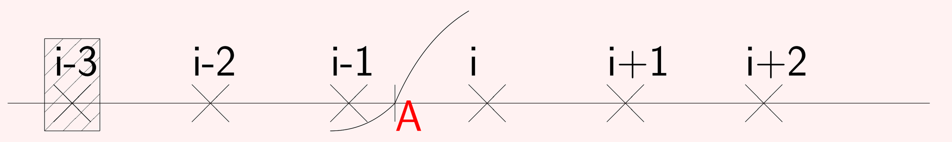
Objectives: at least first order accuracy on κ.
Issues: how to solve accurately and quickly the eikonal equation? When is the reinitialization step necessary?

Reinitialization algorithms

• Relaxation algorithms with subcell fix [4, 5]
Resolution of the unsteady eikonal equation (with fictitious time step τ)

$$\partial_\tau \phi + \mathrm{sgn}(\phi_0)\left(|\nabla \phi| - 1\right) = 0,$$

with $\phi(\tau = 0) := \phi_0$. When steady state is reached, ϕ is the signed distance to $\Gamma = \{\phi_0 = 0\}$.
RK3-TVD in pseudo-time, WENO-5 scheme with subcell fix in space



standard WENO-5 stencil for $D_x^- \phi_i$ is $(x_{i-3}, x_{i-2}, \cdots, x_{i+2})$.
Subcell fix stencil is $(x_{i-2}, x_{i-1}, x_A, x_i, x_{i+1}, x_{i+2})$: use of an ENO-3 scheme ⇒ avoids important displacement of Γ.

⇝ **third order accuracy** on ϕ, first order accuracy on κ, but computational cost $O(N^2)$.

• Second order fast sweeping algorithm [3]
Idea: Gauss-Seidel like iterations to solve the eikonal equation. First order [6] or **second order** [3] methods might be designed with computational cost $O(N)$.
Main issue: needs an accurate initialization near the interface.

• Hybrid method Use relaxation method in a narrow band B_n of Γ, until third order accuracy is achieved. Use fast sweeping elsewhere. Typically,

$$B_n = \{d(x, \Gamma) < 5h\}.$$

- Global second order accuracy on ϕ,
- Third order accuracy near Γ,
- First order accuracy on κ,
- computational cost $O(N^{1+\alpha})$, with $\alpha < 0.5$.

Coupling strategy for evolving interfaces

- Introduce $r_g := \| |\nabla \phi| - 1 \|$ and a threshold δ,
- *initialization:* start with ϕ the signed distance function,
- *transport:* as long as $r_g < \delta$, evolve ϕ according to

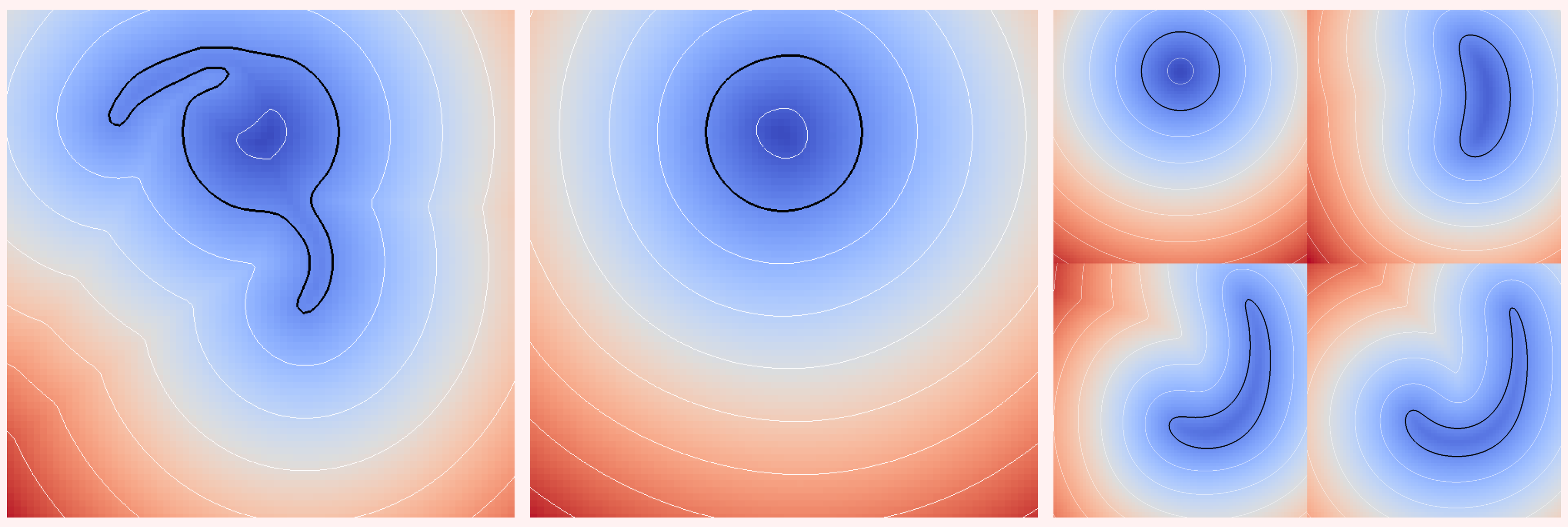
$$\partial_t \phi + \mathbf{U} \cdot \nabla \phi = 0,$$

- *reinitialization:* if $r_g \geq \delta$, replace ϕ by the distance function; go to transport.

Pros:
- **third order convergence for ϕ, first order convergence for κ,**
- ϕ remains close to a distance function ⇒ lower amplitudes of the error on κ,
- the method is **insensitive to Δt** (for $\delta > 0$).

Cons:
- choice of δ ($\delta = 0$ corresponds to usual strategies),
- full computation of the signed distance function.

Numerical illustration

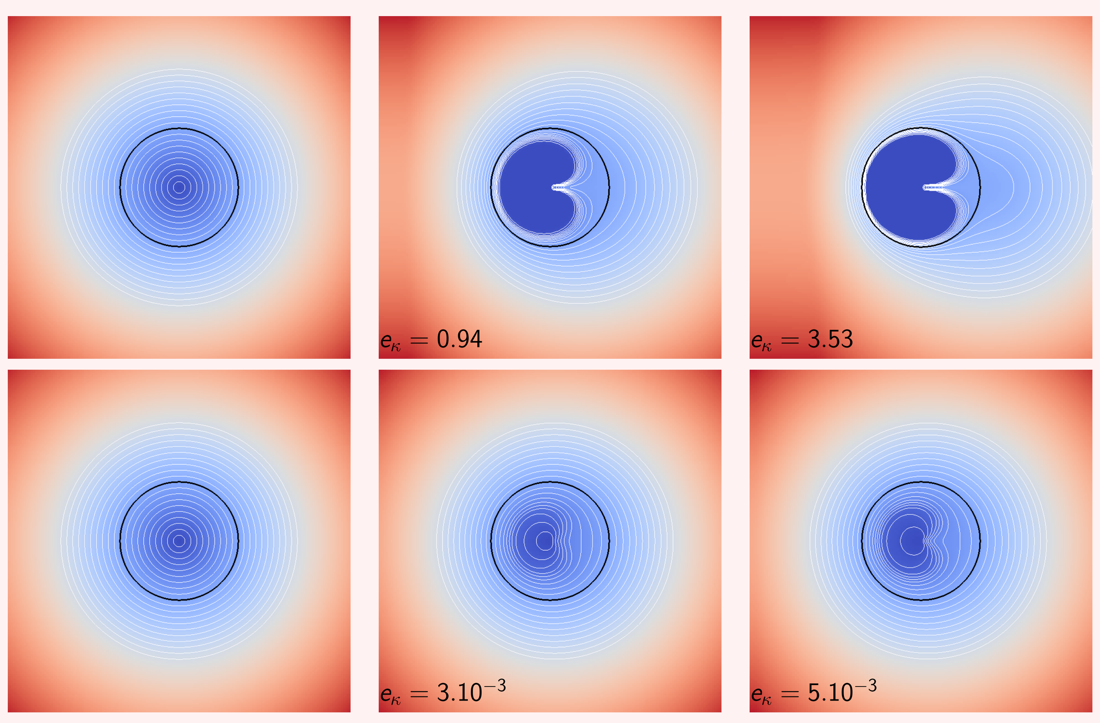


Vortex test case: a circle is distorted and restored at its initial position. Iso contours of ϕ at the end of the simulation ($t = 4$). Left: with $\delta = 0$, which corresponds to usual strategy; middle: with $\delta = 0.1$; right: distortion of ϕ at times $t = 0,\ 2/3,\ 4/3,\ 2$.

The number of reinitialization steps has to be limited.

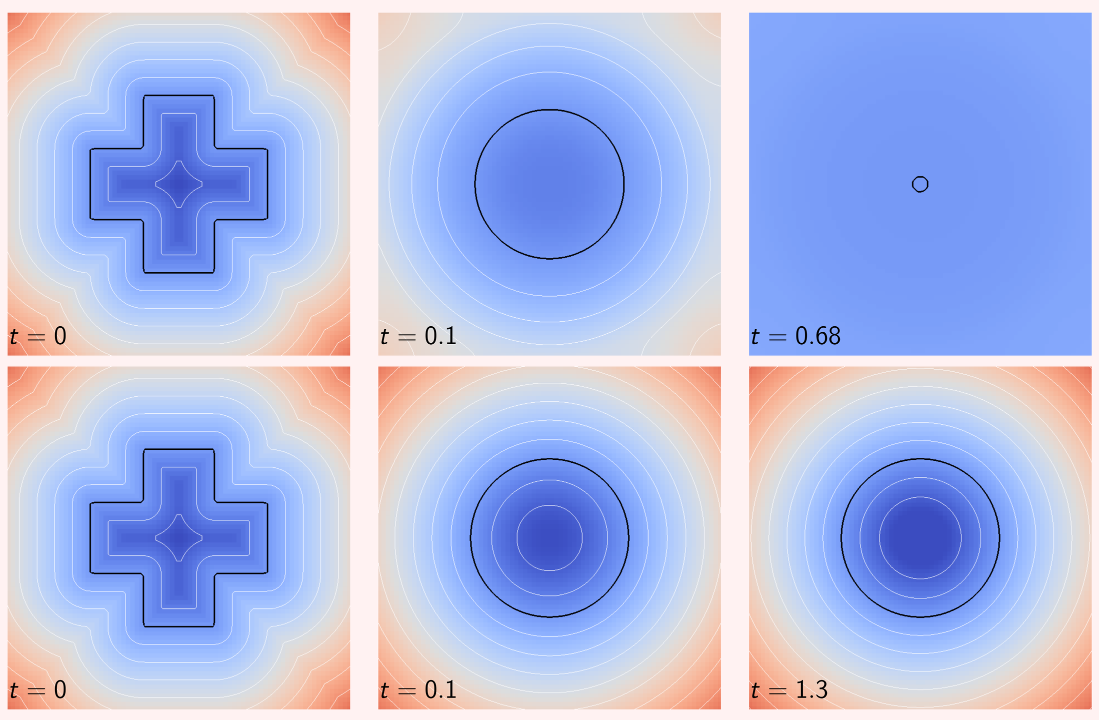
Numerical validation

Creation of large and/or small gradients



Example of a flow around a cylinder. Γ is globally invariant. (top) without reinitialization, with $t = 0,\ 3,\ 6$. (bottom) with $\delta = 0.1$ at the same times. Error on κ is given.

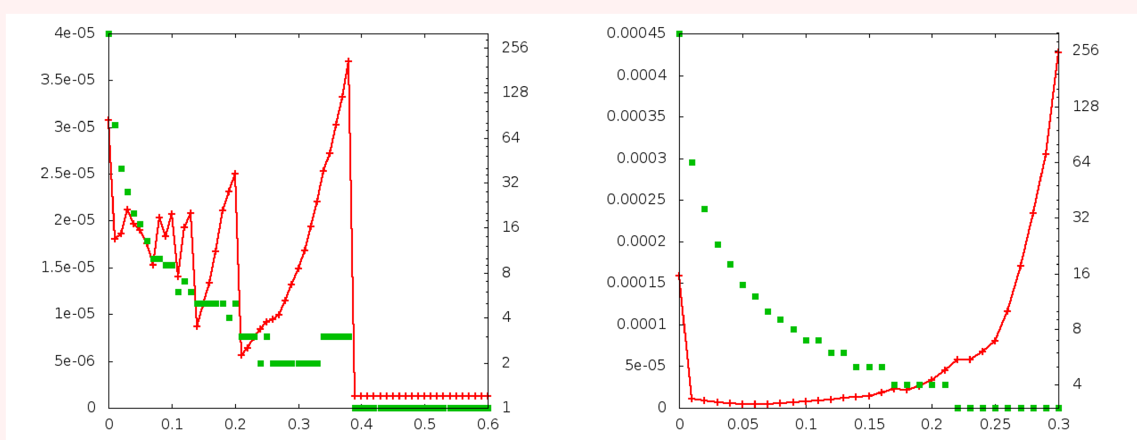
Interface evolving with curvature dependent speed: $\partial_t \phi = (\kappa - \bar{\kappa})|\nabla \phi|$



(top) without reinitialization. (bottom) with $\delta = 0.1$. $\bar{\kappa}$ is the mean value of κ on Γ

Influence of the parameter δ

$\delta = 0$	usual level strategy
$\delta >> 1$	no reinitialization



Vortex test case (left) and flow around cylinder (right): error on the position of Γ depending on δ. The squares represent the total number of reinitialization steps required (logarithmic scale).
⇒ Reinitialization is necessary but the number of reinitializations has to be controlled.

Conclusions and prospects

- Reinitialization steps are often necessary, but their number has to be limited,
- We have designed a reinitialization algorithm with a good balance between accuracy (high order near the interface) and computational cost (fast sweeping step)
- The strategy may be extended for 3D cases, and parallelization is achievable
- The method may be used for passively advected interfaces, interfaces moving with curvature dependent speed, two phase flows with important effect of the surface tension...
- A sharp cartesian method for NS equation might be achieved, with application for air-water interfaces and fluid/solid interactions (wave breaking, marine engineering, wave energy converters...)
- It can serve as a basis to automatically build overset meshes in the vicinity of the interface (e.g. in fluid/solid interactions to capture boundary layers)

References

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