A new Lagrange finite element method for Maxwell equations

A. Bonito [1], J.-L. Guermond [1], F. Luddens [1,2]

[1]Department of Mathematics,
Texas A& M University
College Station, TX

[2]Université Paris XI
Orsay, France

Finite Element Rodeo, March 5-6, 2010

Position of the problem

Objectives

Given a domain Ω, solve the eigenvalue problem :

$$\left\{ \begin{array}{rcll} \nabla\times\nabla\times\mathbf{E} & = & \lambda\mathbf{E} & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

Position of the problem

Objectives

Given a domain Ω, solve the eigenvalue problem :

$$\begin{cases} \nabla\times\nabla\times\mathbf{E} & = & \lambda\mathbf{E} & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} & = & 0 & \text{on } \partial\Omega \end{cases}$$

Requirements

- use Lagrange finite element

Position of the problem

Objectives

Given a domain Ω, solve the eigenvalue problem :

$$\begin{cases} \nabla\times\nabla\times\mathbf{E} = \lambda\mathbf{E} & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

Requirements

- use Lagrange finite element
- use low order polynomials

Position of the problem

Objectives

Given a domain Ω, solve the eigenvalue problem :

$$\begin{cases} \nabla\times\nabla\times\mathbf{E} = \lambda\mathbf{E} & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

Requirements

- use Lagrange finite element
- use low order polynomials
- use as less as possible information about Ω

Boundary value problem

First consider, for $\mathbf{E} \in \mathbf{H}$ the following

$$\left\{ \begin{array}{c} \text{find } \mathbf{B} \in \mathbf{X} \text{ such that} \\ \nabla\times\nabla\times\mathbf{B} = \mathbf{E} \end{array} \right.$$

with :

$$\begin{aligned} \mathbf{H} &:= \left\{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla\cdot\mathbf{F} = 0\right\} \\ \mathbf{X} &:= \mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H} \\ \mathbf{H}_{0,\mathrm{curl}}(\Omega) &:= \left\{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla\times\mathbf{F} \in \mathbf{L}^2(\Omega) \text{ and } \mathbf{F}\times\mathbf{n}_{|\partial\Omega} = 0\right\} \end{aligned}$$

Variational problem

Problem

$$\begin{cases} \text{find } \mathbf{B} \in \mathbf{X} \text{ such that } \forall \mathbf{F} \in \mathbf{X} \\ (\nabla\times\mathbf{B}, \nabla\times\mathbf{F}) = (\mathbf{E}, \mathbf{F}) \end{cases}$$

We will write $\mathbf{B} = A\mathbf{E}$.

Variational problem

Problem

$$\begin{cases} \text{find } \mathbf{B} \in \mathbf{X} \text{ such that } \forall \mathbf{F} \in \mathbf{X} \\ (\nabla \times \mathbf{B}, \nabla \times \mathbf{F}) = (\mathbf{E}, \mathbf{F}) \end{cases}$$

We will write $\mathbf{B} = A\mathbf{E}$.

- the bilinear form is coercive on $\mathbf{X}$ $\rightsquigarrow$ A is well-defined.
- we have an eigenvalue problem for A.
- A can be defined on $L^2(\Omega)$.
- we have to deal with the divergence-free constraint.

State of the art

Penalty in $L^2(\Omega)$

- add $(\nabla\cdot A\mathbf{E}, \nabla\cdot\mathbf{F})$ to the bilinear form.

State of the art

Penalty in $L^2(\Omega)$

- add $(\nabla \cdot A\mathbf{E}, \nabla \cdot \mathbf{F})$ to the bilinear form.
- functional framework : $\mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}_{\mathrm{div}}(\Omega)$.
- works well in smooth or convex domains.

State of the art

Penalty in $L^2(\Omega)$

- add $(\nabla\cdot A\mathbf{E}, \nabla\cdot\mathbf{F})$ to the bilinear form.
- functional framework : $\mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}_{\mathrm{div}}(\Omega)$.
- works well in smooth or convex domains.

Non-smooth domains (Costabel et al., '91)

If Ω is non-smooth and non-convex, the space $\mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}^1$ is a closed proper subset of $\mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}_{\mathrm{div}}(\Omega)$.

State of the art

Penalty in $L^2(\Omega)$

- add $(\nabla\cdot A\mathbf{E}, \nabla\cdot\mathbf{F})$ to the bilinear form.
- functional framework : $\mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}_{\mathrm{div}}(\Omega)$.
- works well in smooth or convex domains.

Non-smooth domains (Costabel et al., '91)

If Ω is non-smooth and non-convex, the space $\mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}^1$ is a closed proper subset of $\mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}_{\mathrm{div}}(\Omega)$.

Weighted penalty in $L^2(\Omega)$ (Costabel et al., '02 , Buffa et al., '10)

- add $(w_\gamma \nabla\cdot A\mathbf{E}, w_\gamma \nabla\cdot\mathbf{F})$ to the bilinear form.
- $w_\gamma \sim d^\gamma$, with d =distance to the singular edges/vertices.
- γ depends on the regularity of the domain.

New formulation (I)

Penalty in $\mathbf{H}^{-1}$

Find $A\mathbf{E} \in \mathbf{H}_{0,\text{curl}},\ p \in H^1_0(\Omega)$ s.t., $\forall \mathbf{F} \in \mathbf{H}_{0,\text{curl}},\ q \in H^1_0(\Omega)$,

$$\begin{aligned} (\nabla\times A\mathbf{E}, \nabla\times\mathbf{F}) + (\nabla p, \mathbf{F}) &= (\mathbf{E}, \mathbf{F}) \\ (A\mathbf{E}, \nabla q) - (\nabla p, \nabla q) &= 0 \end{aligned}$$

New formulation (I)

Penalty in $\mathbf{H}^{-1}$

Find $A\mathbf{E} \in \mathbf{H}_{0,\text{curl}},\ p \in H^1_0(\Omega)$ s.t., $\forall \mathbf{F} \in \mathbf{H}_{0,\text{curl}},\ q \in H^1_0(\Omega)$,

$$
\begin{aligned}
(\nabla\times A\mathbf{E}, \nabla\times\mathbf{F}) + (\nabla p, \mathbf{F}) &= (\mathbf{E}, \mathbf{F}) \\
(A\mathbf{E}, \nabla q) - (\nabla p, \nabla q) &= 0
\end{aligned}
$$

Discrete counterpart

Find $A_h\mathbf{E} \in \mathbf{X}_h,\ p_h \in M_h$ such that, for all $\mathbf{F}_h \in \mathbf{X}_h,\ q_h \in M_h$,

$$
\begin{aligned}
& (\nabla\times A_h\mathbf{E}, \nabla\times\mathbf{F}_h) + (\nabla p_h, \mathbf{F}_h) \\
- \quad & (A_h\mathbf{E}, \nabla q_h) + (\nabla p_h, \nabla q_h) \\
= \quad & (\mathbf{E}, \mathbf{F}_h)
\end{aligned}
$$

New formulation (I)

Penalty in $\mathbf{H}^{-1}$

Find $A\mathbf{E} \in \mathbf{H}_{0,\mathrm{curl}},\ p \in H_0^1(\Omega)$ s.t., $\forall \mathbf{F} \in \mathbf{H}_{0,\mathrm{curl}},\ q \in H_0^1(\Omega)$,

$$\begin{aligned}(\nabla\times A\mathbf{E}, \nabla\times\mathbf{F}) + (\nabla p, \mathbf{F}) &= (\mathbf{E}, \mathbf{F})\\ (A\mathbf{E}, \nabla q) - (\nabla p, \nabla q) &= 0\end{aligned}$$

Discrete counterpart

Find $A_h\mathbf{E} \in \mathbf{X}_h,\ p_h \in M_h$ such that, for all $\mathbf{F}_h \in \mathbf{X}_h,\ q_h \in M_h$,

$$\begin{aligned}&(\nabla\times A_h\mathbf{E}, \nabla\times\mathbf{F}_h) + (\nabla p_h, \mathbf{F}_h)\\ -\ &(A_h\mathbf{E}, \nabla q_h) + (\nabla p_h, \nabla q_h)\\ +\ &h^2(\nabla\cdot A_h\mathbf{E}, \nabla\cdot\mathbf{F}_h)\\ =\ &(\mathbf{E}, \mathbf{F}_h)\end{aligned}$$

- the approximation converges to the right solution,
- it works even if the domain is non-smooth and non-convex,
- with stabilization, we can take P_2 elements for $A_h\mathbf{E}$, P_1 for p_h,
- convergence is independent of the degree of the polynomials for M_h.

- the approximation converges to the right solution,
- it works even if the domain is non-smooth and non-convex,
- with stabilization, we can take P_2 elements for $A_h\mathbf{E}$, P_1 for p_h,
- convergence is independent of the degree of the polynomials for M_h.

But we still have compactness issues.

New formulation (II)

Penalty in $\mathbf{H}^{-\alpha}$

Take $\alpha \in \left(\frac{1}{2}, 1\right)$. Find $A_h\mathbf{E} \in \mathbf{X}_h,\ p_h \in M_h$ such that, for all $\mathbf{F}_h \in \mathbf{X}_h,\ q_h \in M_h$,

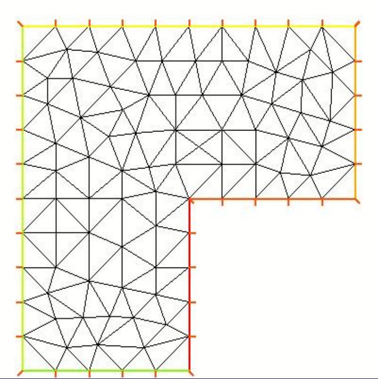
$$\begin{aligned} & (\nabla\times A_h\mathbf{E}, \nabla\times\mathbf{F}_h) + (\nabla p_h, \mathbf{F}_h) \\ - \ & (A_h\mathbf{E}, \nabla q_h) + h^{2(1-\alpha)}(\nabla p_h, \nabla q_h) \\ + \ & h^{2\alpha}(\nabla\cdot A_h\mathbf{E}, \nabla\cdot\mathbf{F}_h) \\ = \ & (\mathbf{E}, \mathbf{F}_h) \end{aligned}$$

Theorem

Consider $A_h : L^2(\Omega) \to L^2(\Omega)$. For $\alpha \in \left(\frac{1}{2}, 1\right)$, the sequence $\{A_h\}_{h>0}$ is collectively compact.

- the approximation converges to the right solution,
- it works even if the domain is non-smooth and non-convex,
- with stabilization, we can take P_2 elements for $A_h\mathbf{E}$, P_1 for p_h,
- convergence is independent of the degree of the polynomials for M_h,

~~But we still have compactness issues.~~



$\lambda_1 \approx 1.47562182408$		
h	val	rel. err
0,1	1.612	8.8E-02
0,05	1.568	6.1E-02
0,025	1.545	4.6E-02
0,0125	1.520	2.9E-02
$\lambda_2 \approx 3.53403136678$		
h	val	rel. err
0,1	3.536	6.5-04
0,05	3.535	1.8E-04
0,025	3.534	4.9E-05
0,0125	3.534	1.4E-05

THANK YOU