Error Estimates for Reduced Order MHD Model based on POD

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Outline of the Talk

Outline of talk:

- Magnetohydrodynamic (MHD) Equations
- Finite Element Galerkin Scheme and Error Estimates
- POD Galerkin Scheme and Error Estimates
- Computational Implementation: Control of MHD
  - Finite element model & optimal control
  - Low dimensional model & optimal control

Model for viscous incompressible electrically conducting flows

- **u**-velocity; *p*-pressure; **B**-magnetic field

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\nabla^2 \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S(\mathrm{curl}\,\mathbf{B}) \times \mathbf{B} = \mathbf{f} \qquad \text{in } \Omega \times (0, T]$$

$$\frac{\partial \mathbf{B}}{\partial t} + \frac{1}{Re_m}\mathrm{curl}\,(\mathrm{curl}\,\mathbf{B}) - \mathrm{curl}\,(\mathbf{u} \times \mathbf{B}) = \mathrm{curl}\,\mathbf{j} \qquad \text{in } \Omega \times (0, T]$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0 \qquad \text{in } \Omega \times (0, T]$$

- Boundary values:
  $\mathbf{u} = \mathbf{0}\,, \quad \mathbf{B}\cdot\mathbf{n} = 0 \quad \text{and} \quad (\mathrm{curl}\,\mathbf{B}) \times \mathbf{n} = 0 \qquad \text{on } \partial\Omega \times (0, T]$
- Initial values: $\mathbf{u}(\mathbf{x}, 0) = \mathbf{0} \quad \text{and} \quad \mathbf{B}(\mathbf{x}, 0) = \mathbf{0} \quad \text{in} \quad \Omega\,,$

- *Re*- Reynolds number; Re_m- Magnetic Reynolds number;
  S- coupling parameter

Functional Framework

Divergence free spaces

$$\mathbf{H}_u = \mathbf{H}_B := \{\mathbf{w} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{w} = 0 \quad \text{and} \quad \mathbf{w} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega\},$$

$$\mathbf{V}_u := \mathbf{H}_u \cap \mathbf{H}_0^1(\Omega) \;\text{ and }\; \mathbf{V}_B := \mathbf{H}_B \cap \mathbf{H}_n^1(\Omega).$$

$\mathbf{H} := \mathbf{H}_u \times \mathbf{H}_B$ and $\mathbf{V} := \mathbf{V}_u \times \mathbf{V}_B$ with inner-products

$$(\chi_1, \chi_2)_{\mathbf{H}} := \int_\Omega \mathbf{u}_1 \cdot \mathbf{u}_2 \, \mathrm{d}\mathbf{x} + S \int_\Omega \mathbf{B}_1 \cdot \mathbf{B}_2 \, \mathrm{d}\mathbf{x}$$

$$(\chi_1, \chi_2)_{\mathbf{V}} := \frac{1}{Re} \int_\Omega \nabla \mathbf{u}_1 : \nabla \mathbf{u}_2 \, \mathrm{d}\mathbf{x} + \frac{S}{Re_m} \int_\Omega \mathrm{curl}\, \mathbf{B}_1 \cdot \mathrm{curl}\, \mathbf{B}_2 \, \mathrm{d}\mathbf{x}$$

Poincare type inequality:

$$\eta \|(\mathbf{u}, \mathbf{B})\|_{\mathbf{H}}^2 \le \|(\mathbf{u}, \mathbf{B})\|_{\mathbf{V}}^2 \quad \forall (\mathbf{u}, \mathbf{B}) \in \mathbf{V}, \quad \text{where} \quad \eta := \min\left\{\frac{\eta_u}{Re}, \frac{\eta_B}{Re_m}\right\}.$$

Weak Form of MHD Equations

Weak form:

$$(\frac{\partial \chi}{\partial t}, \mathbf{w})_{\mathbf{H}} + a(\chi, \mathbf{w}) + b(\chi, \chi, \mathbf{w}) = (\mathbf{F}, \mathbf{w})_{\mathbf{H}}, \quad \forall \chi \in \mathbf{V}$$

Here bilinear form

$$a(\chi_1, \chi_2) = \frac{1}{Re} \int_\Omega \nabla \mathbf{u}_1 : \nabla \mathbf{u}_2 \, \mathrm{d}\mathbf{x} + \frac{S}{Re_m} \int_\Omega \mathrm{curl}\, \mathbf{B}_1 \cdot \mathrm{curl}\, \mathbf{B}_2 \, \mathrm{d}\mathbf{x}$$

and tri-linear form

$$\begin{aligned} b(\chi_1, \chi_2, \chi_3) &= \int_\Omega (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_2 \cdot \mathbf{u}_3 \, \mathrm{d}\mathbf{x} - S \int_\Omega \mathrm{curl}\, \mathbf{B}_2 \times \mathbf{B}_1 \cdot \mathbf{u}_3 \, \mathrm{d}\mathbf{x} \\ &\quad - S \int_\Omega \mathbf{u}_2 \times \mathbf{B}_1 \cdot \mathrm{curl}\, \mathbf{B}_3 \, \mathrm{d}\mathbf{x} \\ &\qquad \forall \chi_i = (\mathbf{u}_i, \mathbf{B}_i) \in \mathbf{V} \end{aligned}$$

Note $\| \cdot \|_{\mathbf{V}} = (a(\cdot, \cdot))^{\frac{1}{2}}$

Properties of Bilinear and Tri-Linear Forms

- $a(\cdot,\cdot)$ is continuous and coercive on **V**:

$$(i)\ |a(\chi_1,\chi_2)| \le 2\|\chi_1\|_{\mathbf{V}}\|\chi_2\|_{\mathbf{V}} \quad \forall \chi_1\,,\chi_2 \in \mathbf{V}\,,$$
$$(ii)\ |a(\chi,\chi)| = \|\chi\|_{\mathbf{V}}^2 \qquad \forall \chi \in \mathbf{V}\,.$$

- $b(\cdot,\cdot,\cdot)$ is skew-symmetric:

$$b(\chi_1,\chi_2,\chi_3) = -b(\chi_1,\chi_3,\chi_2) \quad \forall \chi_1 \in \mathbf{H} \ \text{ and } \ \chi_2,\chi_3 \in \mathbf{V}\,,$$

- $b(\cdot,\cdot,\cdot)$ is continuous:

$$\begin{aligned} |b(\chi_1,\chi_2,\chi_3)| \quad &\le c_a\|\chi_1\|_{\mathbf{H}}^{\frac{1}{2}}\|\chi_1\|_{\mathbf{V}}^{\frac{1}{2}}\|\chi_2\|_{\mathbf{V}}\|\chi_3\|_{\mathbf{H}}^{\frac{1}{2}}\|\chi_3\|_{\mathbf{V}}^{\frac{1}{2}} \\ &+c_b\|\chi_1\|_{\mathbf{H}}^{\frac{1}{2}}\|\chi_1\|_{\mathbf{V}}^{\frac{1}{2}}\|\chi_3\|_{\mathbf{V}}\|\chi_2\|_{\mathbf{H}}^{\frac{1}{2}}\|\chi_2\|_{\mathbf{V}}^{\frac{1}{2}} \qquad \forall \chi_i \in \mathbf{V} \end{aligned}$$

for some constants $c_a, c_b > 0$.

Finite Element Glerkin Scheme and Error Estimate

- Divergence-free finite element space $\mathbf{V}^h \subset \mathbf{V}$. Let $\chi_n^h \approx \chi^h(t_n)$.
- Fully implicit backward Euler in time scheme: seek $\chi_n^h \in \mathbf{V}^h$ such that

$$(\frac{\chi_h^n - \chi_h^{n-1}}{k}, \mathbf{w}_h)_{\mathbf{H}} + a(\chi_h^n, \mathbf{w}_h) + b(\chi_h^n, \chi_h^n, \mathbf{w}_h) = (\mathbf{F}^n, \mathbf{w}_h)_{\mathbf{H}}, \forall \mathbf{w}_h \in \mathbf{V}^h$$

- Appriori estimate:

$$\|\chi_h^n\|_{\mathbf{H}}^2 + k\sum_{i=1}^{n} \|\chi_h^i\|_{\mathbf{V}}^2 \leq k\eta^2 \sum_{i=1}^{n} \|\mathbf{F}^i\|_{\mathbf{H}}^2$$

- Finite element error estimate:

$$\|\chi - \chi_h^n\|_{\mathbf{H}} \leq C(\sigma^{-1}(t_n)k + h^p)$$

POD method

- Given ensemble: $\chi_1^h, \dots, \chi_\ell^h \in \mathbf{V}^h$
- Set $\mathcal{V} := \text{span}\{\chi_1^h, \dots, \chi_\ell^h\} \subset \mathbf{V}^h, \quad m := \dim\mathcal{V} \le \ell$
- Let $\{\psi_i\}_{i=1}^m$ denote orthonormal basis for $\mathcal{V}$
- Each member of ensemble

$$\chi_j^h = \sum_{i=1}^{m} (\chi_j^h, \psi_i)_{\mathbf{v}} \psi_i \quad \text{for } j = 1, \dots, \ell$$

- POD method: Find $d \le m$ orthonormal vectors $\{\psi_i\}_{i=1}^d$ in $\mathbf{V}^h$ minimizing

$$\mathcal{J}(\psi_1, \dots, \psi_d) = \frac{1}{\ell} \sum_{j=1}^{\ell} \left\| \chi_j^h - \sum_{i=1}^{d} (\chi_j^h, \psi_i)_{\mathbf{v}} \psi_i \right\|_{\mathbf{v}}^2$$

subject to

$$(\psi_i, \psi_j)_{\mathbf{V}} = \delta_{ij}$$

Eigenvalue problem

- Necessary conditions of optimality:

$$\mathcal{K}\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad \text{for} \quad i = 1, \ldots, d$$

here $\mathcal{K}_{ij} = \frac{1}{\ell}(\chi_i^h, \chi_j^h)_{\mathbf{v}}$ and $\mathcal{K} \in \mathrm{IR}^{\ell \times \ell}$.

- $\mathcal{K}$ is positive semi-definite and has eigenvalues

$$\lambda_1 \geq \ldots \geq \lambda_m > 0$$

- POD basis of rank $d \leq m$ is given by

$$\psi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{\ell} (\mathbf{v}_i)_j \chi_j^h \ \text{ for } \ i = 1, \ldots, d$$

POD error formula

- Orthogonal projector onto $\mathbf{V}^d = \mathrm{span}\{\psi_i\}_{i=1}^d$:

$$P^d\phi := \sum_{i=1}^{d} (\phi, \psi_i)_{\mathbf{V}} \psi_i \quad \text{for} \quad \phi \in \mathbf{V}^h$$

- $$\mathcal{J}(\psi_1, \ldots, \psi_d) = \frac{1}{\ell} \sum_{j=1}^{\ell} \left\| \chi_j^h - \sum_{i=1}^{d} (\chi_j^h, \psi_i)_{\mathbf{V}} \psi_i \right\|_{\mathbf{V}}^2$$

  Since $\chi_j^h = \sum_{i=1}^m (\chi_j^h, \psi_i)_{\mathbf{V}} \psi_i \,, \quad j = 1, \ldots, \ell,$

$$\frac{1}{\ell} \sum_{j=1}^{\ell} \left\| \chi_j^h - P^d \chi_j^h \right\|_{\mathbf{V}}^2 = \frac{1}{\ell} \sum_{j=1}^{\ell} \sum_{i=d+1}^{m} |(\chi_j^h, \psi_i)_{\mathbf{V}}|^2 = \sum_{j=d+1}^{m} \lambda_i$$

  so that P^d denotes the Ritz projection, i.e., for $\phi \in \mathbf{V}^h$

$$a(P^d\phi, \psi) = a(\phi, \psi) \quad \text{for all} \quad \psi \in \mathbf{V}^d.$$

POD Glerkin Scheme/ Reduced Order Model (ROM)

- Fully implicit in time: Seek $\chi_d^n \in \mathbf{V}^d \subset \mathbf{V}^h$ such that

$$\left(\frac{\chi_d^n - \chi_d^{n-1}}{k}, \mathbf{w}_d\right)_{\mathbf{H}} + a(\chi_d^n, \mathbf{w}_d) + b(\chi_d^n, \chi_d^n, \mathbf{w}_d) = (\mathbf{F}^n, \mathbf{w}_d)_{\mathbf{H}}, \forall \mathbf{w}_d \in \mathbf{V}_d$$

- Appriori estimate:

$$\|\chi_d^n\|_{\mathbf{H}}^2 + k \sum_{i=1}^{n} \|\chi_d^i\|_{\mathbf{V}}^2 \leq k\eta^2 \sum_{i=1}^{n} \|\mathbf{F}^i\|_{\mathbf{H}}^2$$

- Total error: Since $\chi \in \mathbf{V}$, $\chi_h^n \in \mathbf{V}^h$ and $\chi_d^n \in \mathbf{V}_d$,

$$\begin{aligned} \chi - \chi_d^n &= (\chi - \chi_h^n) + (\chi_h^n - \chi_d^n) \\ &= \mathbf{E}_n + \mathbf{e}_n \qquad = \text{FE error} + \text{ROM error} \end{aligned}$$

- ROM error:

$$\begin{aligned} \mathbf{e}_n &= (\chi_h^n - P^d \chi_h^n) + (P^d \chi_h^n - \chi_d^n) \\ &= \alpha_n + \beta_n \end{aligned}$$

Reduced Order Model Error

- ROM error equation:

$$(\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{v}_d)_{\mathbf{H}} + ka(\mathbf{e}_n, \mathbf{v}_d) + k[b(\chi_h^n, \chi_h^n, \mathbf{v}_d) - b(\chi_d^n, \chi_d^n, \mathbf{v}_d)] = \mathbf{0}\,.$$

Take $\mathbf{v}_d = \boldsymbol{\beta}_n$ and note that

(I) Since $a(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n) = 0$ by Ritz projection condition,

$$a(\mathbf{e}_n, \boldsymbol{\beta}_n) = a(\mathbf{e}_n, \mathbf{e}_n) - a(\boldsymbol{\alpha}_n, \boldsymbol{\alpha}_n) = \|\mathbf{e}_n\|_{\mathbf{V}}^2 - \|\boldsymbol{\alpha}_n\|_{\mathbf{V}}^2$$

(II)

$$\begin{aligned} |b(\chi_h^n, \chi_h^n, \boldsymbol{\beta}_n) & - b(\chi_d^n, \chi_d^n, \boldsymbol{\beta}_n)| \\ & = |b(\mathbf{u}_d^n, \mathbf{e}_n, \boldsymbol{\alpha}_n) + b(\mathbf{e}_n, \mathbf{u}_n^h, \boldsymbol{\alpha}_n) - b(\mathbf{e}_n, \mathbf{u}_h^n, \mathbf{e}_n)| \\ & \le \tfrac{1}{2}\|\mathbf{e}_n\|_{\mathbf{V}}^2 + C_0\|\boldsymbol{\alpha}_n\|_{\mathbf{V}}^2 + C_1\|\mathbf{e}_n\|_{\mathbf{H}}^2 \end{aligned}$$

by Holder inequality followed by Sobolve and Young's inequalities.

(III) $(\mathbf{e}_n - \mathbf{e}_{n-1}, \boldsymbol{\beta}_n)_{\mathbf{H}} = (\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{e}_n)_{\mathbf{H}} - (\mathbf{e}_n - \mathbf{e}_{n-1}, \boldsymbol{\alpha}_n)_{\mathbf{H}}$

- $(\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{e}_n)_{\mathbf{H}} = \frac{1}{2}\|\mathbf{e}_n\|_{\mathbf{H}}^2 - \frac{1}{2}\|\mathbf{e}_{n-1}\|_{\mathbf{H}}^2 + \frac{1}{2}\|\mathbf{e}_n - \mathbf{e}_{n-1}\|_{\mathbf{H}}^2$
- $(\mathbf{e}_n - \mathbf{e}_{n-1}, \boldsymbol{\alpha}_n)_{\mathbf{H}} \leq \frac{1}{2}\|\mathbf{e}_n - \mathbf{e}_{n-1}\|_{\mathbf{H}}^2 + \frac{1}{2}\|\boldsymbol{\alpha}_n\|_{\mathbf{H}}^2$

Employing these estimates in the ROM error equation yields:

$$\|\mathbf{e}_n\|_{\mathbf{H}}^2 - \|\mathbf{e}_{n-1}\|_{\mathbf{H}}^2 + k\|\mathbf{e}_n\|_{\mathbf{V}}^2 \leq C_0 k\|\boldsymbol{\alpha}_n\|_{\mathbf{v}}^2 + C_1 k\|\mathbf{e}_n\|_{\mathbf{H}}^2 + \|\boldsymbol{\alpha}_n\|_{\mathbf{H}}^2$$

Summing for $n = 1$ to ℓ yields

$$\frac{1}{\ell}\|\mathbf{e}_\ell\|_{\mathbf{H}}^2 + \frac{1}{\ell}\sum_{n=1}^{\ell} k\|\mathbf{e}_n\|_{\mathbf{v}}^2 \leq \frac{1}{\ell}\sum_{n=1}^{\ell}\|\boldsymbol{\alpha}_n\|_{\mathbf{H}}^2 + C_0 k\frac{1}{\ell}\sum_{n=1}^{\ell} k\|\boldsymbol{\alpha}_n\|_{\mathbf{v}}^2 + C_1 k\frac{1}{\ell}\sum_{n=1}^{\ell}\|\mathbf{e}_n|_{\mathbf{H}}^2$$

- Assuming k satisfies $C_1 k \leq \theta < 1$ and applying Discrete Gronwall Inequality yields

$$\frac{1-\theta}{\ell}\|\mathbf{e}_\ell\|_{\mathbf{H}}^2 + \frac{k}{l}\sum_{n=1}^{\ell}\|\mathbf{e}_n\|_{\mathbf{V}}^2 \leq e^{\theta}(\frac{1}{\eta} + C_0 k) \sum_{n=d+1}^{m} \lambda_n$$

- Error estimate:

$$\begin{aligned} \|\chi - \chi_d^n\|_{\mathbf{H}} &\leq \|\chi - \chi_h^n\|_{\mathbf{H}} + \|\chi_h^n - \chi_d^n\|_{\mathbf{H}} \\ &\leq C(\sigma^{-1}(t_n)k + h^p) + \left[\tfrac{\ell}{1-\theta} e^{\theta}(\tfrac{1}{\eta} + C_0 k)\textstyle\sum_{n=d+1}^{m}\lambda_n\right]^{\frac{1}{2}} \end{aligned}$$

Semi-implicit Backward Euler (Scheme II):

Error equation:

$$(\frac{\chi_d^n - \chi_d^{n-1}}{k}, \mathbf{w}_d)_{\mathbf{H}} + a(\chi_d^n, \mathbf{w}_d) + b(\chi_d^{n-1}, \chi_d^n, \mathbf{w}_d) = (\mathbf{F}^n, \mathbf{w}_d)_{\mathbf{H}}\ , \forall \mathbf{w}_d \in \mathbf{V}_d$$

We similarly obtain

$$\|\mathbf{e}_n\|_{\mathbf{H}}^2 - \|\mathbf{e}_{n-1}\|_{\mathbf{H}}^2 + \frac{5k}{4}\|\mathbf{e}_n\|_{\mathbf{V}}^2 \le \frac{k}{4}\|\mathbf{e}_{n-1}\|_{\mathbf{V}}^2 + C_0 k\|\alpha_n\|_{\mathbf{v}}^2 + C_2 k\|\mathbf{e}_{n-1}\|_{\mathbf{H}}^2 + \|\alpha_n\|_{\mathbf{H}}^2$$

Summing from $n = 1$ to ℓ and applying Discrete Gronwall inequality yields

$$\frac{1}{\ell}\|\mathbf{e}_\ell\|_{\mathbf{H}}^2 + \frac{k}{l}\sum_{n=1}^{\ell}\|\mathbf{e}_n\|_{\mathbf{v}}^2 \le e^{C_2 k}(\frac{1}{\eta} + C_0 k)\sum_{n=d+1}^{m}\lambda_n$$

Error estimate (for Scheme II):

$$\|\chi - \chi_d^n\|_{\mathbf{H}} \le C(\sigma^{-1}(t_n)k + h^p) + \left[\ell e^{C_2 k}\left(\frac{1}{\eta} + C_0 k\right)\sum_{n=d+1}^{m}\lambda_n\right]^{\frac{1}{2}}$$

Goal: Drive a velocity/magnetic field to a desired one by applied current

- Optimal control problem: Find the minimizer $(\boldsymbol{\chi}, \mathrm{curl}\,\mathbf{j}) \in \mathbf{V}$ for

$$\min_{(\boldsymbol{\chi}, \mathrm{curl}\,\mathbf{j})} \mathscr{J}(\boldsymbol{\chi}, \mathrm{curl}\,\mathbf{j}) := \int_0^T \frac{\alpha}{2}\|\boldsymbol{\chi} - (\mathbf{u}^d, \mathbf{B}^d)\|^2 \,\mathrm{d}t + \frac{\beta}{2}\|\mathrm{curl}\,\mathbf{j}\|\|^2 \,\mathrm{d}t,$$

  subject to

$$(\frac{\partial \boldsymbol{\chi}}{\partial t}, \mathbf{w})_{\mathbf{H}} + a(\boldsymbol{\chi}, \mathbf{w}) + b(\boldsymbol{\chi}, \boldsymbol{\chi}, \mathbf{w}) = ((\mathbf{f}, \mathrm{curl}\,\mathbf{j}), \mathbf{w})_{\mathbf{H}}\,, \quad \forall \boldsymbol{\chi} \in \mathbf{V}$$

- Finite Element Approximation: Find $(\boldsymbol{\chi}^h, \mathrm{curl}\,\mathbf{j}^h) \in \mathbf{V}^h$ that minimizes $\mathscr{J}^h(\boldsymbol{\chi}^h, \mathrm{curl}\,\mathbf{j}^h)$ subject to Finite Element model

- Reduced Order Approximation: Find $(\boldsymbol{\chi}^d, \mathrm{curl}\,\mathbf{j}^d) \in \mathbf{V}^d$ that minimizes $\mathscr{J}^d(\boldsymbol{\chi}^d, \mathrm{curl}\,\mathbf{j}^d)$ subject to Reduced Order Model

- State (MHD) equation:

$$(\frac{\partial \chi}{\partial t}, \mathbf{w})_{\mathbf{H}} + a(\chi, \mathbf{w}) + b(\chi, \chi, \mathbf{w}) = (\mathbf{F}, \mathbf{w})_{\mathbf{H}}\,, \quad \forall \chi \in \mathbf{V}$$

$$\chi(\mathbf{x}, 0) = \chi_0$$

- Adjoint equation for adjoint state $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$:

$$-(\frac{\partial \xi}{\partial t}, \mathbf{w})_{\mathbf{H}} + a(\xi, \mathbf{w}) + b(\chi, \mathbf{w}, \xi) + b(\mathbf{w}, \chi, \xi) = (\alpha(\mathbf{u} - \mathbf{u}^d, \mathbf{B} - \mathbf{B}^d), \mathbf{w})_{\mathbf{H}}\,,$$

$$\xi(\mathbf{x}, T) = \mathbf{0}$$

- Optimality condition: $\mathrm{curl}\,\mathbf{j} = -\frac{1}{\beta}\boldsymbol{\xi}_2$
- Challenge: CPU and memory intensive ... needs fast/realtime algorithm (e.g. SQP/Newton methods)
- Computationa strategy: Combine fast optimization methods with model reduction for PDE

- Parameters: $\Delta t = 0.005$, $h = \frac{1}{20}$, $Re = 10000$, $Re_m = 100$ and $S = 10$
- Initial velocity field:

$$\mathbf{u}_0(x,y) = (\cos(2\pi y)(\cos(2\pi x) - 1), \sin(2\pi x)\sin(2\pi y))$$

(two rotating vortices)

- Initial magnetic field:

$$\mathbf{B}_0(x,y) = (\sin(\pi x)\cos(\pi y), -\cos(\pi x)\sin(\pi y))$$

(single vortex rotating counter-clockwise).

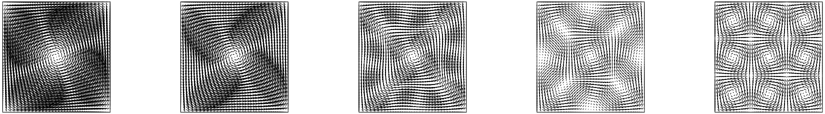
- Target velocity field:

$$u^d(x,y) = \sin(2\pi x)\sin(2\pi y)\,, \quad v^d(x,y) = \cos(2\pi x)(\cos(2\pi y) - 1)\,,$$

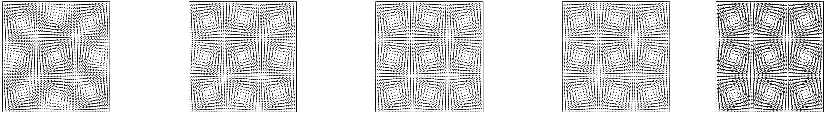
(two vortices)

- Target magnetic field: (six vortices)

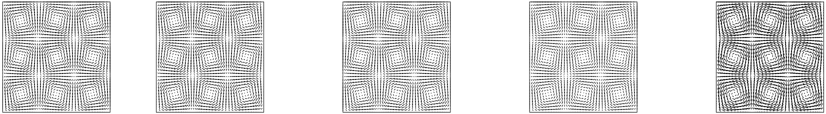
$$b_1^d(x,y) = \sin(3\pi x)\cos(3\pi y)\,, \quad b_2^d(x,y) = -\cos(3\pi x)\sin(3\pi y)\,.$$



(a) t=0.0 (b) t=0.01 (c) t=0.02 (d) t=0.03 (t_m) target $\mathbf{B}^d$

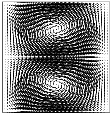


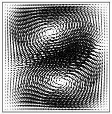
(e) t=0.04 (f) t=0.05 (g) t=0.06 (h) t=0.07 (t_m) target $\mathbf{B}^d$

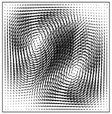


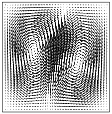
(i) t=0.07 (j) t=0.08 (k) t=0.09 (l)t=0.1 (t_m) target $\mathbf{B}^d$

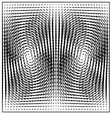
Optimal controlled magnetic field $\mathbf{B}$ at various time instants along with the target field $\mathbf{B}^d$ (fifth column)

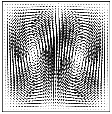

(a) t=0.0

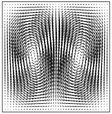

(b) t=0.01

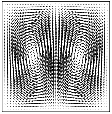

(c) t=0.02

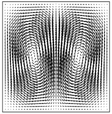

(d) t=0.03

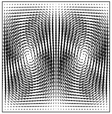

(t_u) target $\mathbf{u}^d$

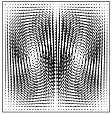

(e) t=0.04

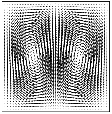

(f) t=0.05

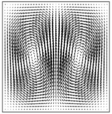

(g) t=0.06

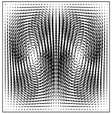

(h) t=0.07

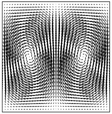

(t_u) target $\mathbf{u}^d$


(i) t=0.07


(j) t=0.08


(k) t=0.09


(l)t=0.1


(t_m) target $\mathbf{u}^d$

Controlled velocity field $\mathbf{u}$ at time instants along with target (fifth column)

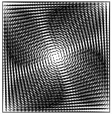
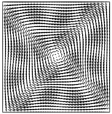
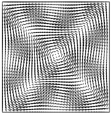
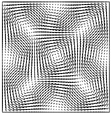
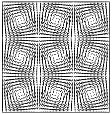
- Snapshots: $\{\chi_h^n\}_{n=1}^{1000}$
- Measures for choosing number of modes d:

$$\mathcal{E} := \frac{\sum_{i=1}^{d} \lambda_i}{\sum_{i=1}^{m} \lambda_i} \times 100\%; \qquad \widehat{\mathcal{E}} := \frac{\max_{1 \le n \le N} \|\chi_h^n - \chi_d^n\|}{\max_{1 \le n \le N} \|\chi_h^n\|} \times 100\%$$

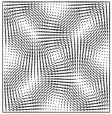
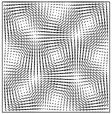
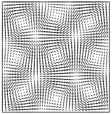
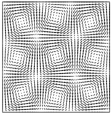
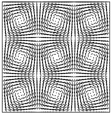
- Choice between **H**-norm and **V**-norm in POD:

	m=2	m=6	m=10	m=14	m=18
$\mathcal{E}$, **H** norm	59.68	79.83	89.92	93.96	99.9
$\mathcal{E}$, **V** norm	50.06	70.49	80.74	88.86	92.93
$\widehat{\mathcal{E}}$, **H** norm	52.8	39.83	27.92	13.96	3.8
$\widehat{\mathcal{E}}$, **V** norm	53.6	40.49	28.74	18.86	8.4

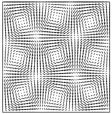
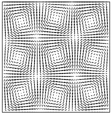
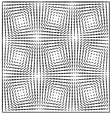
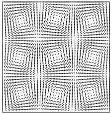
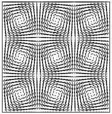
- CPU time savings: FE requires 42 times as much CPU time as ROM

(a) t=0.0 (b) t=0.01 (c) t=0.02 (d) t=0.03 (t_m) target $\mathbf{B}^d$

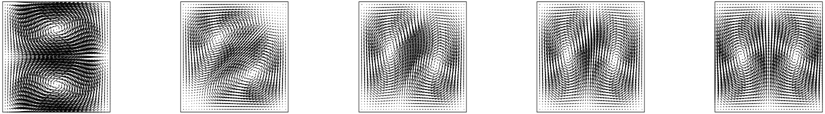






(e) t=0.04 (f) t=0.05 (g) t=0.06 (h) t=0.07 (t_m) target $\mathbf{B}^d$

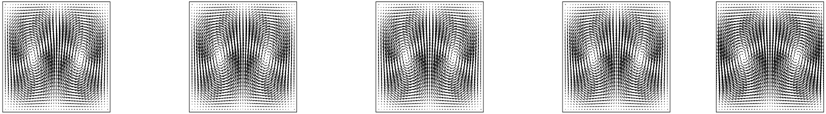






(i) t=0.07 (j) t=0.08 (k) t=0.09 (l)t=0.1 (t_m) target $\mathbf{B}^d$

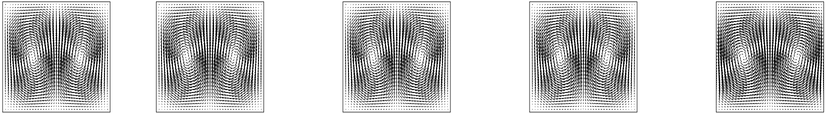
Reduced order controlled magnetic field **B** at time instants along with target (fifth column)



(a) t=0.0 (b) t=0.01 (c) t=0.02 (d) t=0.03 (t_u) target $\mathbf{u}^d$



(e) t=0.04 (f) t=0.05 (g) t=0.06 (h) t=0.07 (t_u) target $\mathbf{u}^d$



(i) t=0.07 (j) t=0.08 (k) t=0.09 (l)t=0.1 (t_m) target $\mathbf{u}^d$

Reduced order controlled velocity field $\mathbf{u}$ at time instants along with target (fifth column)