# A POD based non-linear observer for unsteady flows

**Edoardo Lombardi**
*MAB* - Université Bordeaux I, France

**Jessie Weller**
*MAB* - Université Bordeaux I, France

**Marcelo Buffoni**
*AFM* - University of Southampton, UK

**Angelo Iollo**
*MAB* - Université Bordeaux I and INRIA Project MC2, France

# Summary

1. Motivation
2. Low-dimensional modeling of unsteady flows
   (a) Low-order model construction
   (b) Low-order model with feedback control construction
3. A non-linear state observer for unsteady flows
   (a) Non-linear observer
   (b) Results
      i. Two-dimensional case with feedback control: Re = 150
      ii. Three-dimensional case: Re = 300
4. Analysis of the capabilities with filtering technique
   (a) Filtering technique
   (b) Results for three-dimensional case: Re = 300

# Motivations

- Low-order models gave satisfactory prediction results for laminar 2D flows around bluff bodies and, in particular, for the configuration considered in this work. (Galletti *et al.*, JFM, 2004)
- Typical control tools cannot be applied to Navier-Stokes equations (high number of degrees of freedom in their discretization)
- Compute control laws by Reduced Order Models
- State estimation: recover the entire flow field from a limited number of flow measurements

# Low-order model construction

- Discrete instantaneous velocity expanded in terms of empirical eingenmodes:

$$\boldsymbol{u}(\boldsymbol{x},t) = \overline{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{c}(t)\boldsymbol{u}_c(\boldsymbol{x}) + \sum_{n=1}^{N_r} a_n(t)\boldsymbol{\phi}_n(\boldsymbol{x})$$

where $\boldsymbol{c}(t)$ is the feedback control law and $\overline{\boldsymbol{u}}(\boldsymbol{x})$ and $\boldsymbol{u}_c(\boldsymbol{x})$ are reference velocity fields and chosen such that the snapshots are equal to zero at inflow, outflow and *jet* boundaries.

- Eigenmodes $\phi_n(x)$ are found by proper orthogonal decomposition (POD) using the "***snapshots method***" of Sirovich (1987).
- Limited number of POD modes, $N_r$, is used in the representation of velocity fields (snapshots) $\longrightarrow$ they are the modes giving the main contribution to the flow energy.

# Low-order model construction

- Galerkin projection of the Navier-Stokes equations over the retained POD modes leading to the low-order model:

$$\begin{aligned}
\dot{a}_r(t) &= A_r + C_{kr}a_k(t) - B_{ksr}a_k(t)a_s(t) - E_r\dot{c}(t) - F_r c^2(t) + [G_r - H_{kr}a_k(t)]c(t) \\
a_r(0) &= (\boldsymbol{u}(\boldsymbol{x},0) - \overline{\boldsymbol{u}}(\boldsymbol{x}) - \boldsymbol{c}(0)\boldsymbol{u}_c(\boldsymbol{x}), \boldsymbol{\phi}_r)
\end{aligned}$$

# Low-order model construction

- Galerkin projection of the Navier-Stokes equations over the retained POD modes leading to the low-order model:

$$\begin{aligned}\dot{a}_r(t) &= A_r + C_{kr}a_k(t) - B_{ksr}a_k(t)a_s(t) - E_r\dot{c}(t) - F_r c^2(t) + [G_r - H_{kr}a_k(t)]c(t)\\ a_r(0) &= (\boldsymbol{u}(\boldsymbol{x},0) - \overline{\boldsymbol{u}}(\boldsymbol{x}) - \boldsymbol{c}(0)\boldsymbol{u}_c(\boldsymbol{x}), \boldsymbol{\phi}_r)\end{aligned}$$

- Coefficient $B_{ksr}$ derives directly from the Galerkin projection of the non-linear terms in the Navier-Stokes equations

# Low-order model construction

- Galerkin projection of the Navier-Stokes equations over the retained POD modes leading to the low-order model:

$$
\begin{aligned}
\dot{a}_r(t) &= A_r + C_{kr}a_k(t) - B_{ksr}a_k(t)a_s(t) - E_r\dot{c}(t) - F_r c^2(t) + [G_r - H_{kr}a_k(t)]c(t) \\
a_r(0) &= (\boldsymbol{u}(\boldsymbol{x},0) - \overline{\boldsymbol{u}}(\boldsymbol{x}) - \boldsymbol{c}(0)\boldsymbol{u}_c(\boldsymbol{x}), \boldsymbol{\phi}_r)
\end{aligned}
$$

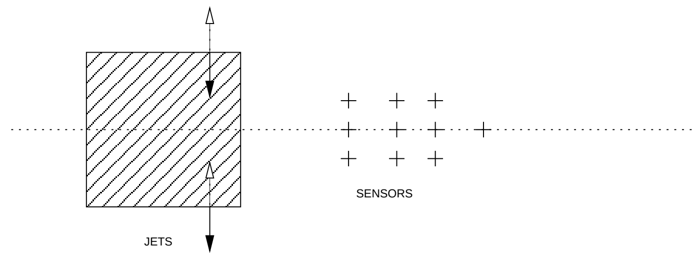
- Coefficient $B_{ksr}$ derives directly from the Galerkin projection of the non-linear terms in the Navier-Stokes equations

- System matrices $A$, $C$, $E$, $F$, $G$ and $H$ are *calibrated* minimizing

$$
\begin{aligned}
\mathcal{J} = & \int_0^T \sum_{r=1}^{N_r} \left(\dot{a}_r(t) - \dot{\hat{a}}_r(t)\right)^2 \mathrm{d}t + \sum_{r=1}^{N_r} \alpha \left(A_r - \hat{A}_r\right)^2 \\
& + \sum_{r=1}^{N_r} \sum_{k=1}^{N_r} \alpha \left(C_{kr} - \hat{C}_{kr}\right)^2 + \sum_{r=1}^{N_r} \alpha \left(E_r - \hat{E}_r\right)^2 \\
& + \sum_{r=1}^{N_r} \alpha \left(F_r - \hat{F}_r\right)^2 + \sum_{r=1}^{N_r} \alpha \left(G_r - \hat{G}_r\right)^2 \\
& + \sum_{r=1}^{N_r} \sum_{k=1}^{N_r} \alpha \left(H_{kr} - \hat{H}_{kr}\right)^2
\end{aligned}
$$

where $\alpha << 1$.

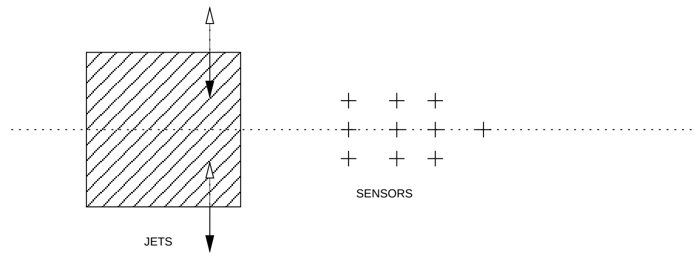
# Low-order model construction with feedback actuation




- Control law can be obtained by feedback, using vertical velocity measurements at points $\boldsymbol{x}_S$ in cylinder wake

$$c(t) = K\boldsymbol{v}(t, \boldsymbol{x}_S)$$

# Low-order model construction with feedback actuation




- Control law can be obtained by feedback, using vertical velocity measurements at points $\boldsymbol{x}_S$ in cylinder wake

  $c(t) = K\boldsymbol{v}(t, \boldsymbol{x}_S)$

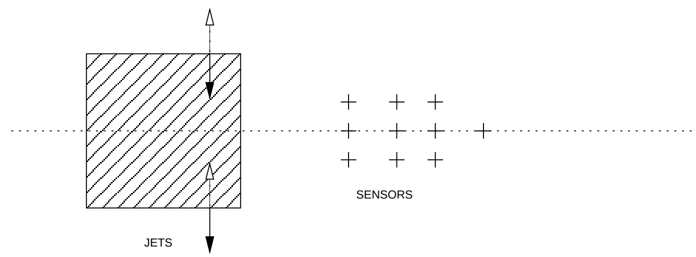
- Developing the velocity at measurements points

$$\boldsymbol{v}(t, \boldsymbol{x}_S) = \overline{\boldsymbol{v}}(\boldsymbol{x}_S) + c(t)\boldsymbol{v}_c(\boldsymbol{x}_s) + \sum_{n=1}^{N_r} a_n(t)\boldsymbol{\phi}_n(\boldsymbol{x}_s)$$

$$\Downarrow$$

$$\boldsymbol{v}(t, \boldsymbol{x}_S) = \overline{\boldsymbol{v}}(\boldsymbol{x}_S) + K\boldsymbol{v}(t, \boldsymbol{x}_S)\boldsymbol{v}_c(\boldsymbol{x}_s) + \sum_{n=1}^{N_r} a_n(t)\boldsymbol{\phi}_n(\boldsymbol{x}_s)$$

# Low-order model construction with feedback actuation




- Control law can be obtained by feedback, using vertical velocity measurements at points $\boldsymbol{x}_S$ in cylinder wake

  $c(t) = K\boldsymbol{v}(t, \boldsymbol{x}_S)$

- Developing the velocity at measurements points

  $$\boldsymbol{v}(t, \boldsymbol{x}_S) = \overline{\boldsymbol{v}}(\boldsymbol{x}_S) + c(t)\boldsymbol{v}_c(\boldsymbol{x}_s) + \sum_{n=1}^{N_r} a_n(t)\boldsymbol{\phi}_n(\boldsymbol{x}_s)$$

  $$\Downarrow$$

  $$\boldsymbol{v}(t, \boldsymbol{x}_S) = \overline{\boldsymbol{v}}(\boldsymbol{x}_S) + K\boldsymbol{v}(t, \boldsymbol{x}_S)\boldsymbol{v}_c(\boldsymbol{x}_s) + \sum_{n=1}^{N_r} a_n(t)\boldsymbol{\phi}_n(\boldsymbol{x}_s)$$

- $\Rightarrow$ Low-order model with feedback control in compact form:

  $\dot{a}_r(t) = A_r^* + C_{kr}^* a_k(t) - B_{ksr}^* a_k(t) a_s(t)$

  where the matrices $A_r^*$, $B_{ksr}^*$ and $C_{kr}^*$ are functions of $K$, $\overline{\boldsymbol{v}}(\boldsymbol{x}_S)$,$\boldsymbol{v}_c(\boldsymbol{x}_s)$ and $\boldsymbol{\phi}_n(\boldsymbol{x}_s)$.

# Non-linear observer

- Galerkin representation of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ in terms of $N_r$ empirical eigenfunctions, $\boldsymbol{\Phi}^i(\boldsymbol{x})$, obtained by Proper Orthogonal Decomposition (POD)

$$\boldsymbol{u}(\boldsymbol{x}, t) = \overline{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{c}(t)\boldsymbol{u}_c(\boldsymbol{x}) + \sum_{i=1}^{N_r} a_i(t)\boldsymbol{\Phi}^i(\boldsymbol{x})$$

# Non-linear observer

- Galerkin representation of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ in terms of $N_r$ empirical eigenfunctions, $\boldsymbol{\Phi}^i(\boldsymbol{x})$, obtained by Proper Orthogonal Decomposition (POD)

$$\boldsymbol{u}(\boldsymbol{x}, t) = \overline{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{c}(t)\boldsymbol{u}_c(\boldsymbol{x}) + \sum_{i=1}^{N_r} a_i(t)\boldsymbol{\Phi}^i(\boldsymbol{x})$$

- Two simple approaches to estimate coefficients $a_i(t)$:

# Non-linear observer

- Galerkin representation of the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ in terms of $N_r$ empirical eigenfunctions, $\boldsymbol{\Phi}^i(\boldsymbol{x})$, obtained by Proper Orthogonal Decomposition (POD)

$$\boldsymbol{u}(\boldsymbol{x},t)=\overline{\boldsymbol{u}}(\boldsymbol{x})+\boldsymbol{c}(t)\boldsymbol{u}_c(\boldsymbol{x})+\sum_{i=1}^{N_r}a_i(t)\boldsymbol{\Phi}^i(\boldsymbol{x})$$

- Two simple approaches to estimate coefficients $a_i(t)$:
  1. LSQ $\Rightarrow$ approximate flow measurements in a least square sense (Galletti *et al.* (2004), Venturi & Karniadakis (2004) and Willcox (2006))

$$a_j(\tau)=\sum_{k=1}^{N_s}\Upsilon_{kj}\left(\boldsymbol{\Phi}(\boldsymbol{x})\right)f_k\left(\boldsymbol{u}\left(\boldsymbol{x},\tau\right)\right)$$

# Non-linear observer

- Galerkin representation of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ in terms of $N_r$ empirical eigenfunctions, $\boldsymbol{\Phi}^i(\boldsymbol{x})$, obtained by Proper Orthogonal Decomposition (POD)

$$\boldsymbol{u}(\boldsymbol{x}, t) = \overline{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{c}(t)\boldsymbol{u}_c(\boldsymbol{x}) + \sum_{i=1}^{N_r} a_i(t)\boldsymbol{\Phi}^i(\boldsymbol{x})$$

- Two simple approaches to estimate coefficients $a_i(t)$:
  1. LSQ $\Rightarrow$ approximate flow measurements in a least square sense (Galletti *et al.* (2004), Venturi & Karniadakis (2004) and Willcox (2006))

$$a_j(\tau) = \sum_{k=1}^{N_s} \Upsilon_{kj}\left(\boldsymbol{\Phi}(\boldsymbol{x})\right) f_k\left(\boldsymbol{u}\left(\boldsymbol{x}, \tau\right)\right)$$

  2. LSE $\Rightarrow$ assume that a linear correlation exists between the flow measurements and the value of the POD modal coefficients

$$a_j(\tau) = \sum_{k=1}^{N_s} \Lambda_{kj}\left(\hat{a}_j, \hat{f}_k\right) f_k\left(\boldsymbol{u}\left(\boldsymbol{x}, \tau\right)\right)$$

# Non-linear observer

- Galerkin representation of the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ in terms of $N_r$ empirical eigenfunctions, $\boldsymbol{\Phi}^i(\boldsymbol{x})$, obtained by Proper Orthogonal Decomposition (POD)

$$\boldsymbol{u}(\boldsymbol{x},t) = \overline{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{c}(t)\boldsymbol{u}_c(\boldsymbol{x}) + \sum_{i=1}^{N_r} a_i(t)\boldsymbol{\Phi}^i(\boldsymbol{x})$$

- Two simple approaches to estimate coefficients $a_i(t)$:
  1. LSQ $\Rightarrow$ approximate flow measurements in a least square sense (Galletti *et al.* (2004), Venturi & Karniadakis (2004) and Willcox (2006))

$$a_j(\tau) = \sum_{k=1}^{N_s} \Upsilon_{kj}\left(\boldsymbol{\Phi}(\boldsymbol{x})\right) f_k\left(\boldsymbol{u}\left(\boldsymbol{x},\tau\right)\right)$$

  2. LSE $\Rightarrow$ assume that a linear correlation exists between the flow measurements and the value of the POD modal coefficients

$$a_j(\tau) = \sum_{k=1}^{N_s} \Lambda_{kj}\left(\hat{a}_j, \hat{f}_k\right) f_k\left(\boldsymbol{u}\left(\boldsymbol{x},\tau\right)\right)$$

- Problems with linear estimation (LSQ and LSE) when 3D flows with complicated unsteady patterns are considered

# Non-linear observer

- Galerkin representation of the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ in terms of $N_r$ empirical eigenfunctions, $\boldsymbol{\Phi}^i(\boldsymbol{x})$, obtained by Proper Orthogonal Decomposition (POD)

$$\boldsymbol{u}(\boldsymbol{x},t) = \overline{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{c}(t)\boldsymbol{u}_c(\boldsymbol{x}) + \sum_{i=1}^{N_r} a_i(t)\boldsymbol{\Phi}^i(\boldsymbol{x})$$

- Two simple approaches to estimate coefficients $a_i(t)$:
  1. LSQ $\Rightarrow$ approximate flow measurements in a least square sense (Galletti *et al.* (2004), Venturi & Karniadakis (2004) and Willcox (2006))

$$a_j(\tau) = \sum_{k=1}^{N_s} \Upsilon_{kj}\left(\boldsymbol{\Phi}(\boldsymbol{x})\right) f_k\left(\boldsymbol{u}\left(\boldsymbol{x},\tau\right)\right)$$

  2. LSE $\Rightarrow$ assume that a linear correlation exists between the flow measurements and the value of the POD modal coefficients

$$a_j(\tau) = \sum_{k=1}^{N_s} \Lambda_{kj}\left(\hat{a}_j, \hat{f}_k\right) f_k\left(\boldsymbol{u}\left(\boldsymbol{x},\tau\right)\right)$$

- Problems with linear estimation (LSQ and LSE) when 3D flows with complicated unsteady patterns are considered

- Contributions in literature aimed to effective sensor placement and extensions of LSE $\Rightarrow$ QSE ( Schmit & Glauser (2005), Cohen *et al.* (2004), Cohen *et al.* (2006), Willcox (2006))

# Non-linear observer

- Minimize the sum of the residuals
  - LSQ case $\Rightarrow$

$$\boldsymbol{\alpha}(t) = \underset{\boldsymbol{a}(t)}{\operatorname{argmin}} \sum_{m=1}^{N_m} \left( \sum_{r=1}^{N_r} R_r^2(\boldsymbol{a}(\tau_m)) + \sum_{r=1}^{N_r} (a_r(\tau_m) - \sum_{k=1}^{N_s} \Upsilon_{kr} f_k \left(\boldsymbol{u}\left(\tau_m\right)\right))^2 \right)$$

  - LSE case $\Rightarrow$

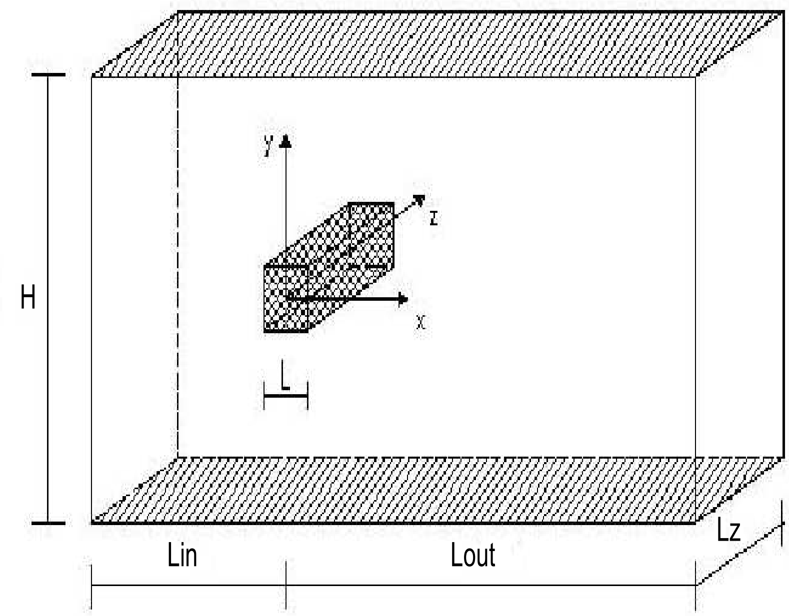
$$\boldsymbol{\alpha}(t) = \underset{\boldsymbol{a}(t)}{\operatorname{argmin}} \sum_{m=1}^{N_m} \left( \sum_{r=1}^{N_r} R_r^2(\boldsymbol{a}(\tau_m)) + \sum_{r=1}^{N_r} (a_r(\tau_m) - \sum_{k=1}^{N_s} \Lambda_{kr} f_k \left(\boldsymbol{u}\left(\tau_m\right)\right))^2 \right)$$

  where $R_r(\boldsymbol{a}(\tau_m))$ is the residual of low-order model
- the method represents a non-linear observer of the flow state (K-LSQ and K-LSE)

# DNS : Computational Domain

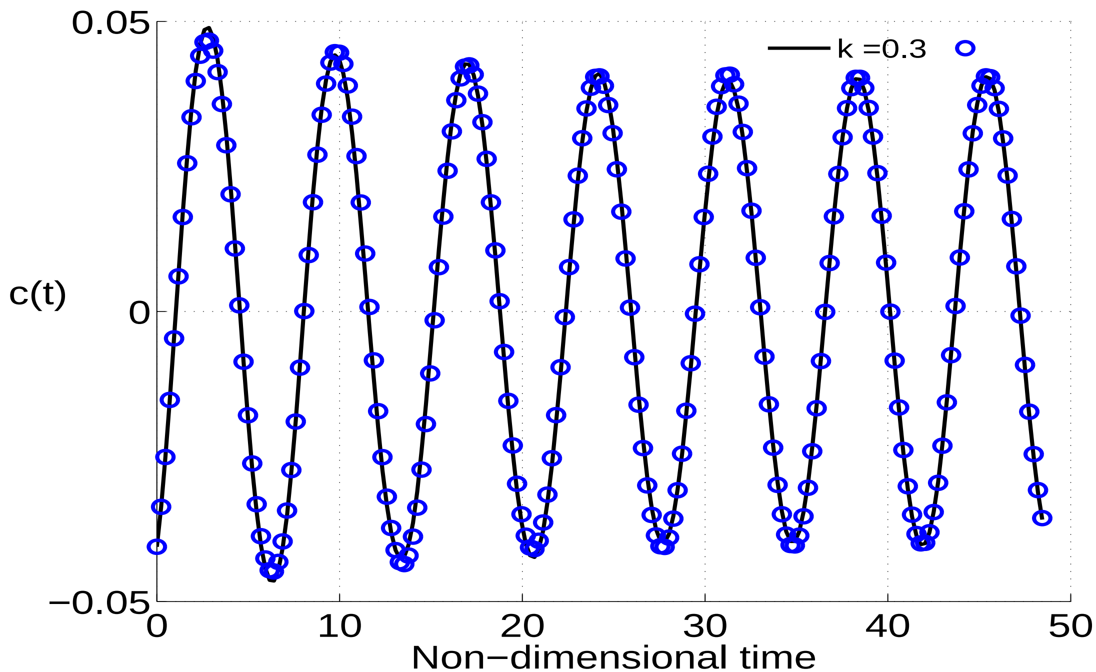
- Dimensions:
  - $L = 1$
  - $H/L = 8$
  - $L_{in}/L = 12$
  - $L_{out}/L = 20$
  - $L_z/L = 0.6$, 2D simulations
  - $L_z/L = 6$, 3D simulations

- Reynolds numbers based on maximum velocity of incoming profile and "L"

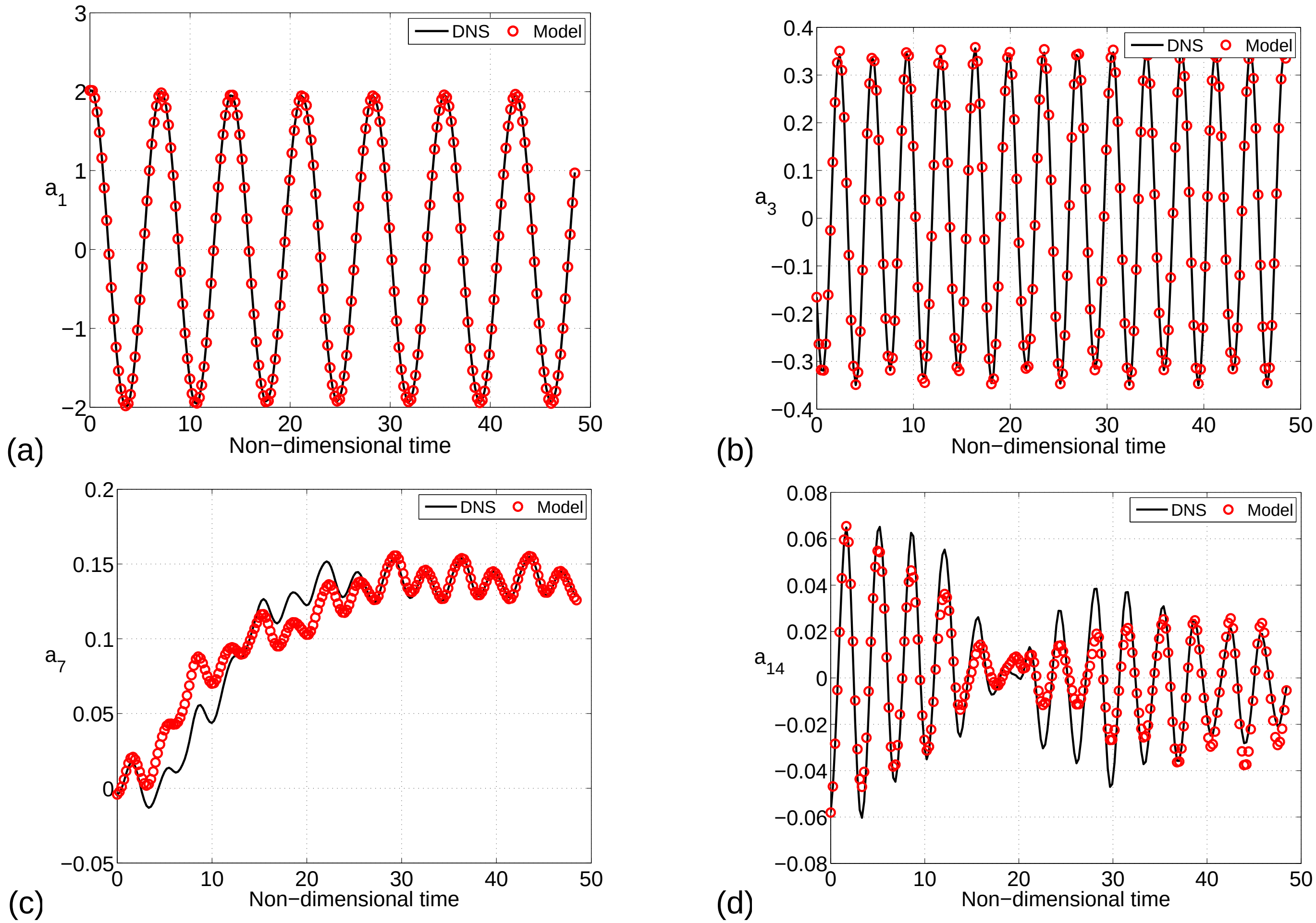


# Observer - Results 2D : POD and ROM set-up

- Database
  - $\approx$ 30 snapshots shedding cycle
  - Re = 150 $\longrightarrow$ 205 snapshots
  - Feedback gain k = 0.3
- Model:
  - 205 snapshots from $t = 0.00$ to $t = 48.46 \longrightarrow \Delta t = 48.46$
  - 20 modes retained $\longrightarrow E = 99.7\%$ with a new control law

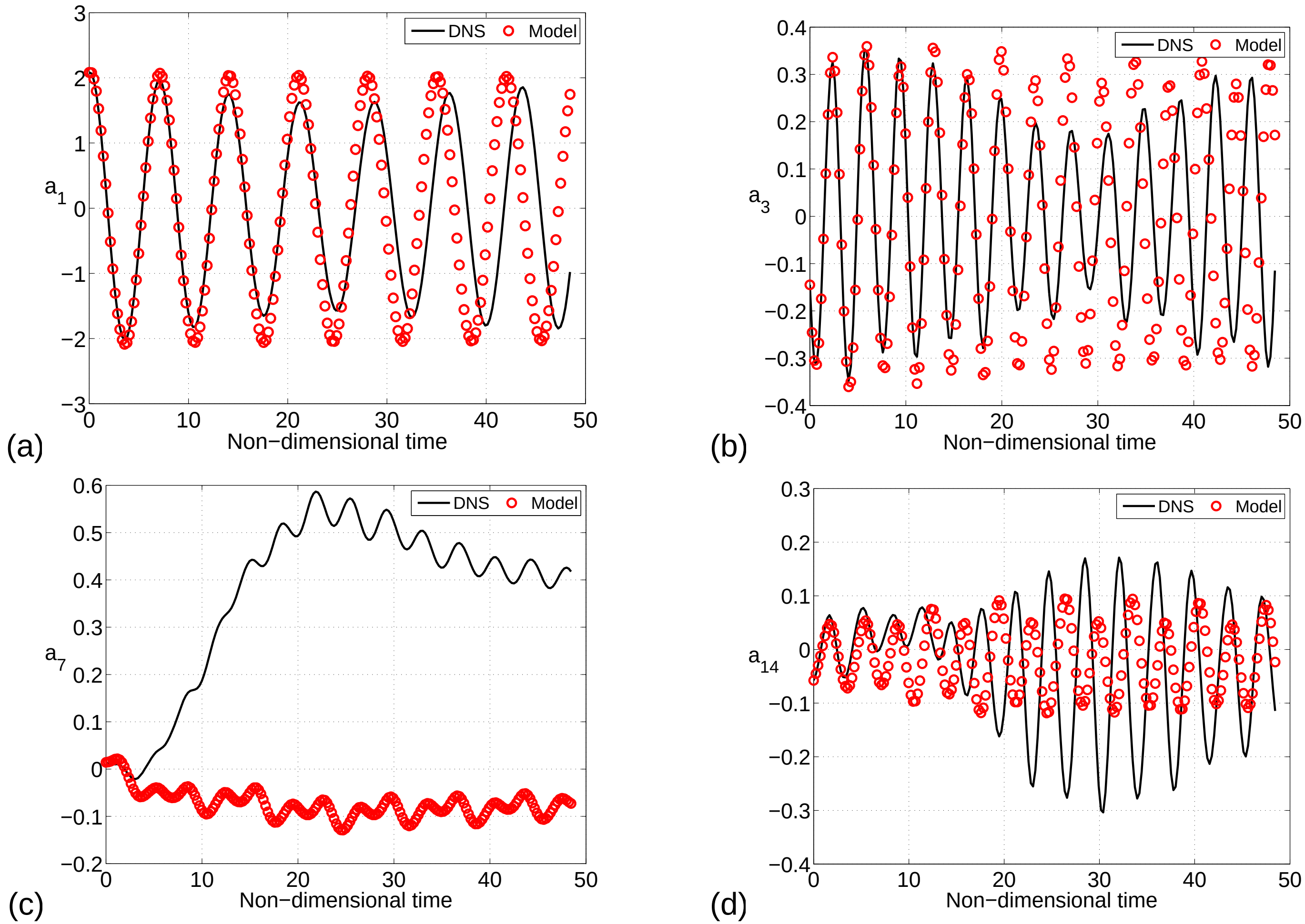


# Results 2D: Modal coefficient predictions $k = 0.3$



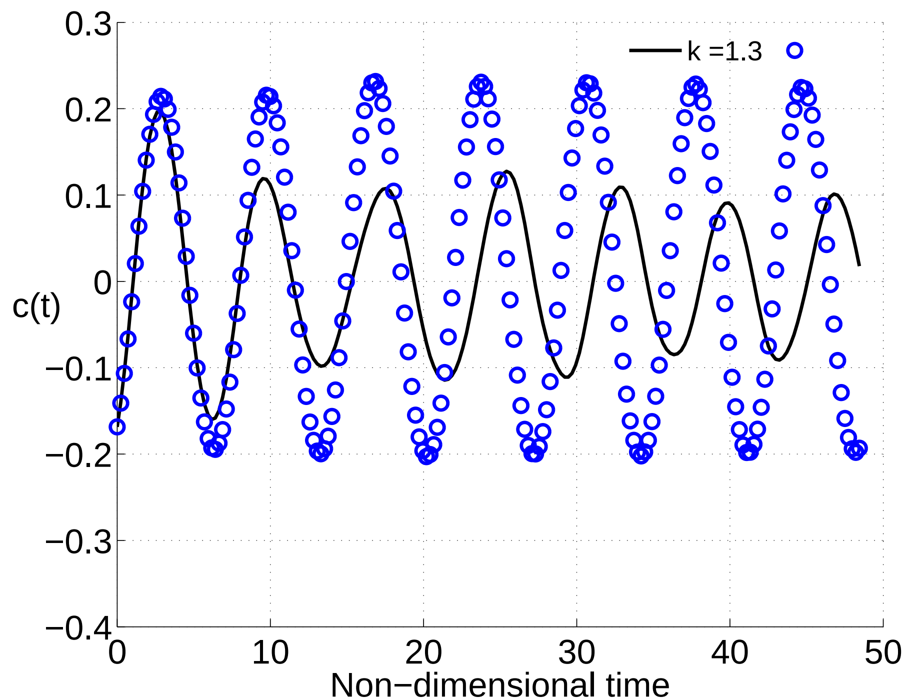

- POD modal coefficients $a_1$,$a_3$,$a_7$ and $a_{14}$. Projection of the fully resolved Navier-Stokes simulations onto POD modes (continuous line) vs. the integration of the dynamical system inside the calibration interval, obtained retaining the first 20 POD modes (circles).

# Results 2D: Modal coefficient predictions $k = 1.3$



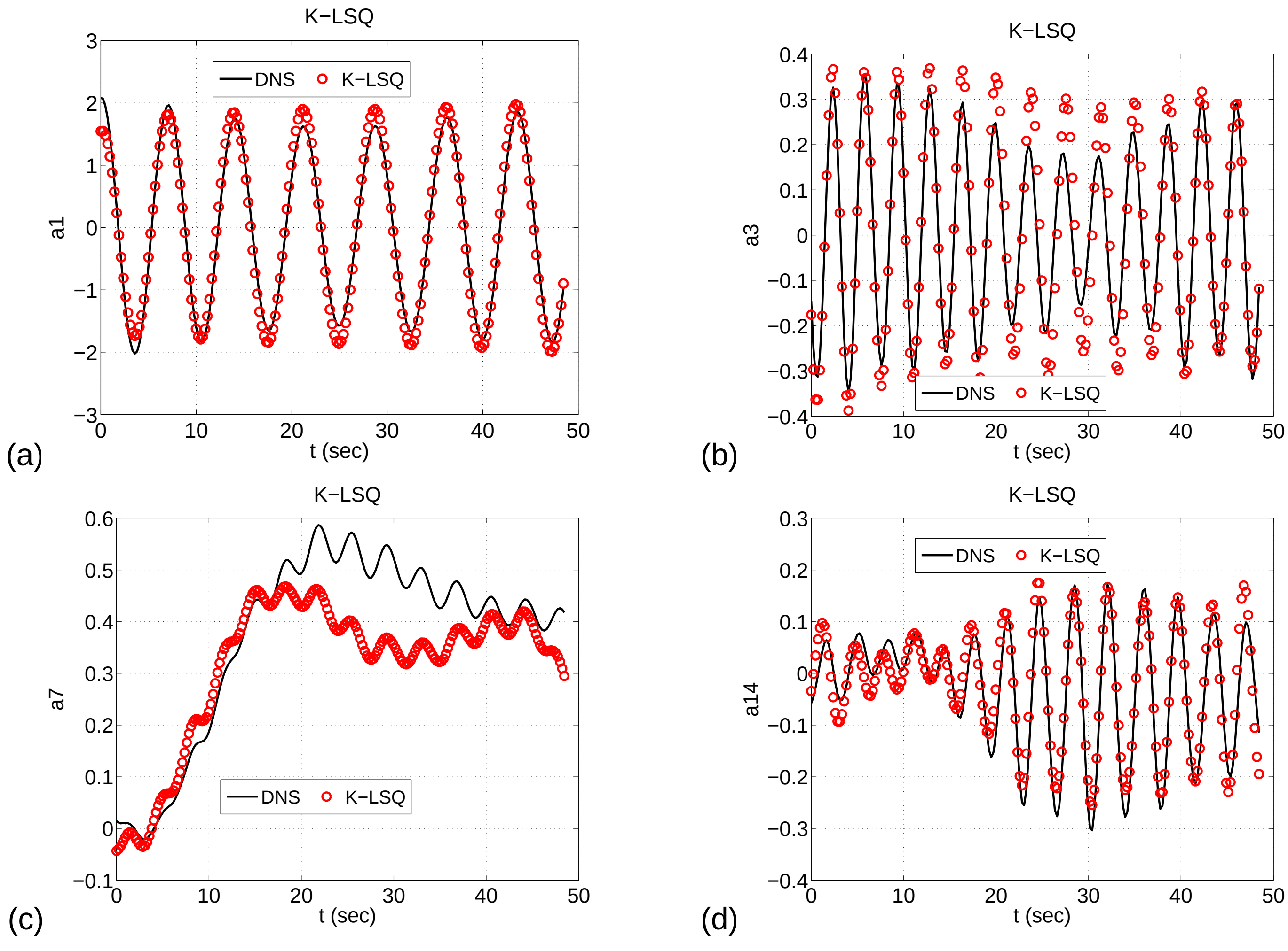

- POD modal coefficients $a_1$,$a_3$,$a_7$ and $a_{14}$. Projection of the fully resolved Navier-Stokes simulations onto POD modes (continuous line) vs. the integration of the dynamical system with a different feedback gain, obtained retaining the first 20 POD modes (circles).

# Results 2D: Control law reconstruction $k = 1.3$



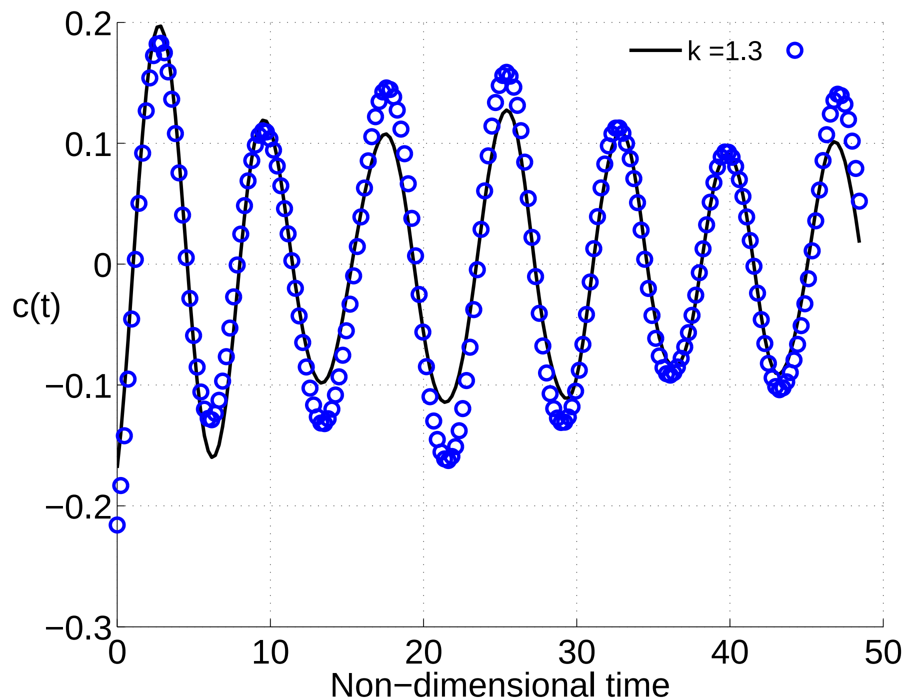

- Projection of the actual control law onto POD modes (continuous line) vs. Reconstructed control law using the integration of the dynamical system with a different feedback gain, obtained retaining the first 20 POD modes (circles).

# Results 2D: KLSQ modal coefficient predictions $k = 1.3$



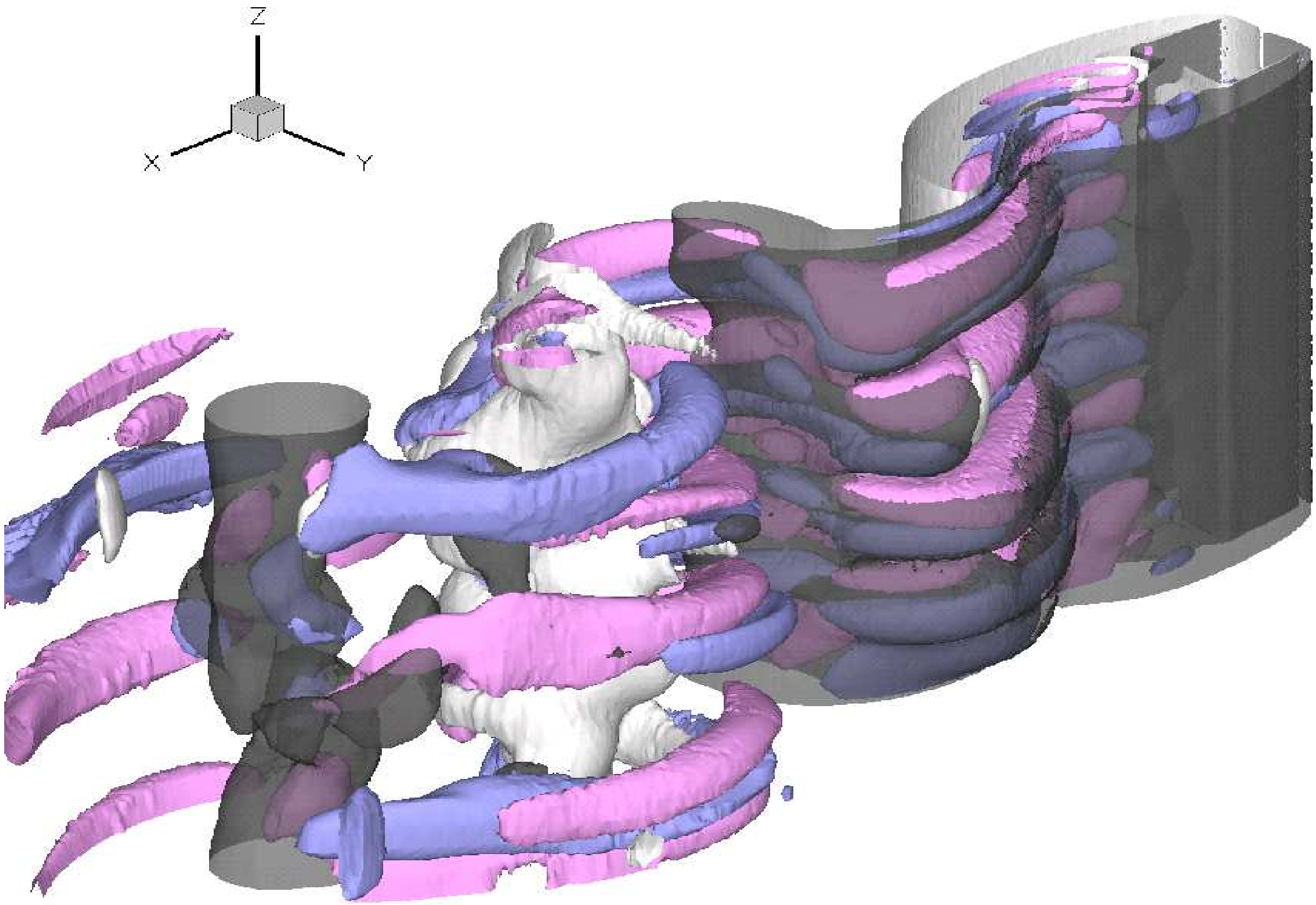

- POD modal coefficients $a_1$,$a_3$,$a_7$ and $a_{14}$. Projection of the fully resolved Navier-Stokes simulations onto POD modes (continuous line) vs. the estimation with the K-LSQ approach (using only six velocity sensors), obtained retaining the first 20 POD modes (circles).

# Results 2D: KLSQ reconstruction $k = 1.3$



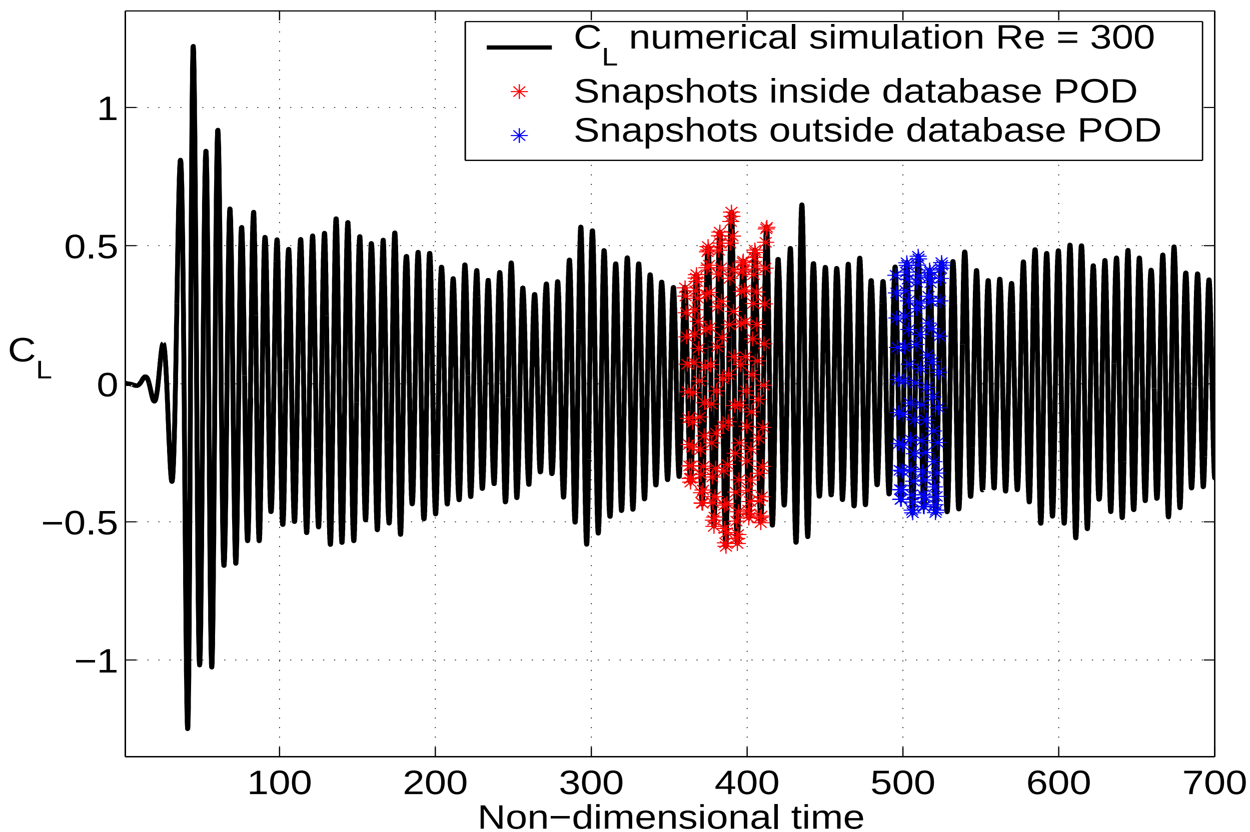

- Projection of the actual control law onto POD modes (continuous line) vs. Reconstructed control law using the the estimation with the K-LSQ approach (using only six velocity sensors),obtained retaining the first 20 POD modes (circles).
- **Actual Flow vs. reconstruction (video)**

# Considered 3D case for low-order modeling: Re = 300

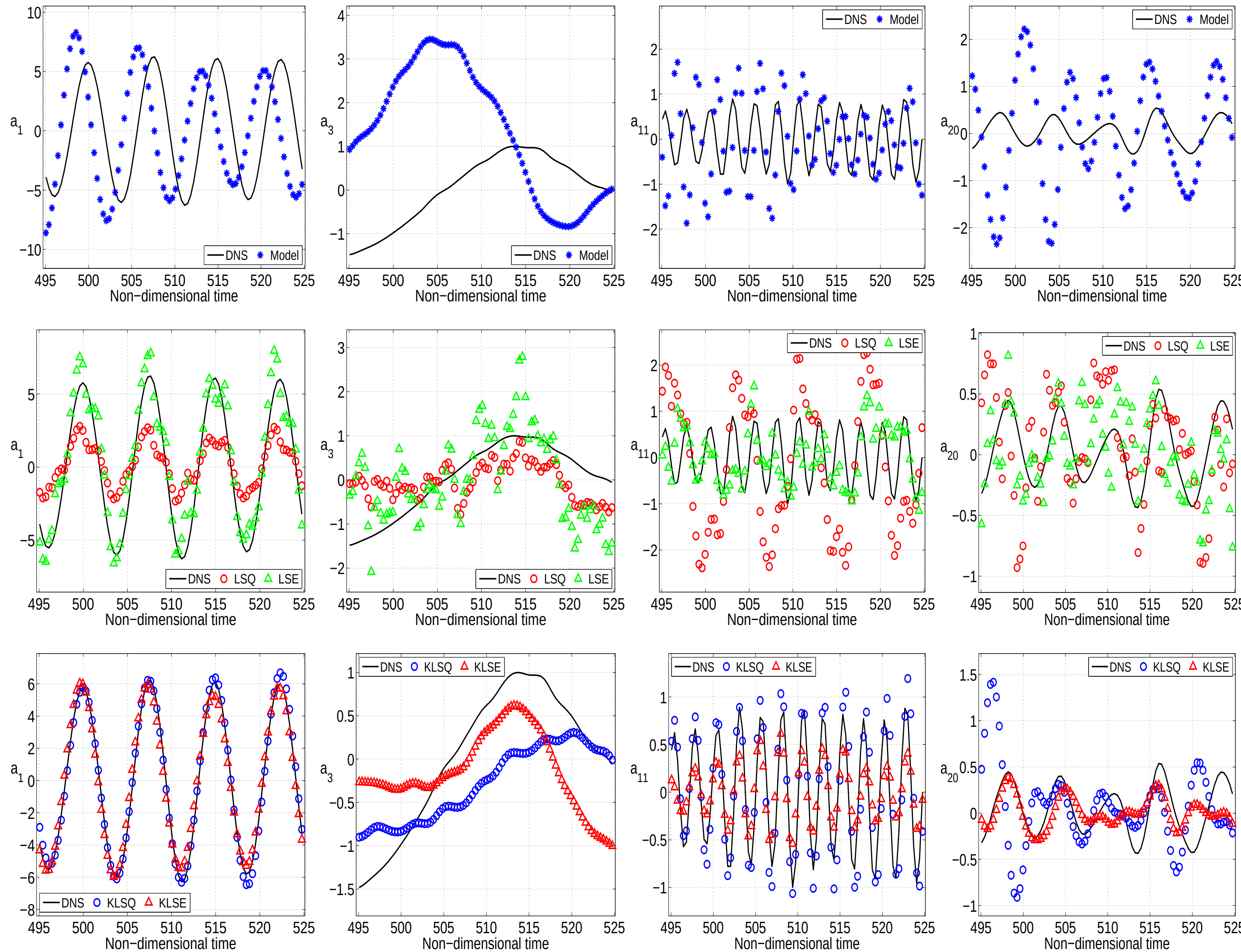


# Results 3D : POD and ROM set-up

- Database
  - $\approx$ 23 snapshots shedding cycle
  - Re = 300 $\longrightarrow$ 1980 snapshots
- Model:
  - POD : 151 snapshots from $t = 360.23$ to $t = 412.64 \longrightarrow \Delta t = 52.41$
  - 20 modes retained $\longrightarrow E = 67.6\%$ outside the database

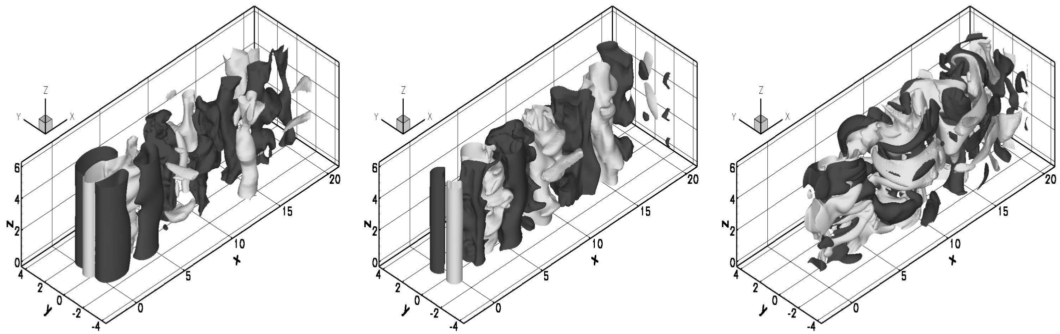


# Results 3D: Modal coefficient predictions

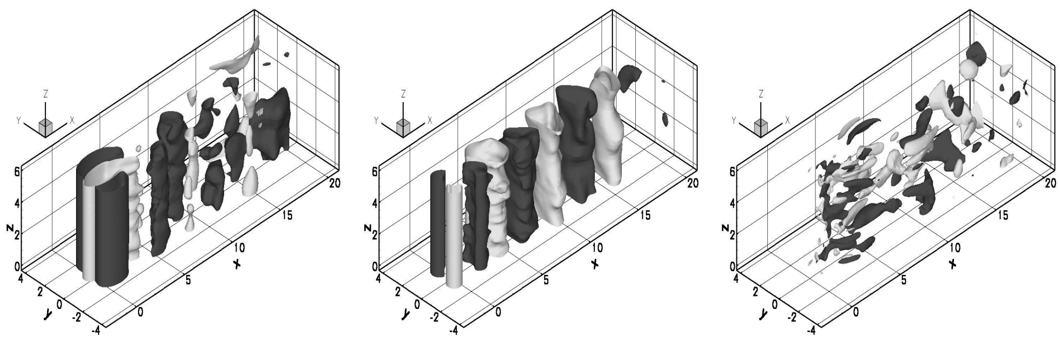



- Some representative modal coefficients estimated vs. DNS projections.
  $1^{st}$ line : POD-ROM ; $2^{nd}$ line : LSQ/LSE ; $3^{rd}$ line : KLSQ/KLSE ($24$ velocity sensors).

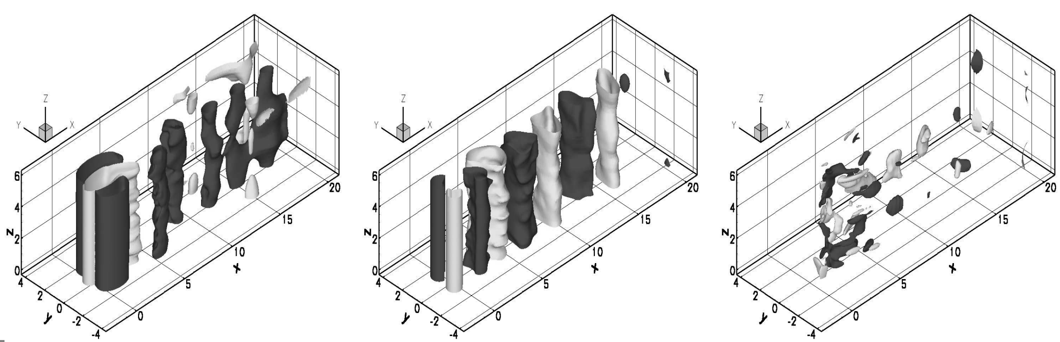
# Results 3D : Flow field estimation



(a)



(b)



(c)

**Isosurfaces of the velocity components of a snapshot outside the database:**

- $u$ (left): grey = 0.5
  dark grey = 1.0
- $v$ (center): grey = -0.25
  dark grey = 0.25
- $w$ (right): grey = -0.075
  dark grey = 0.075
- (a) actual snapshot ($t = 426.6$), (b) snapshot projected on the retained POD modes,(c) reconstructed snapshot using the K-LSE technique
- **Actual Flow vs. Reconstruction (video)**

# Filtering Technique

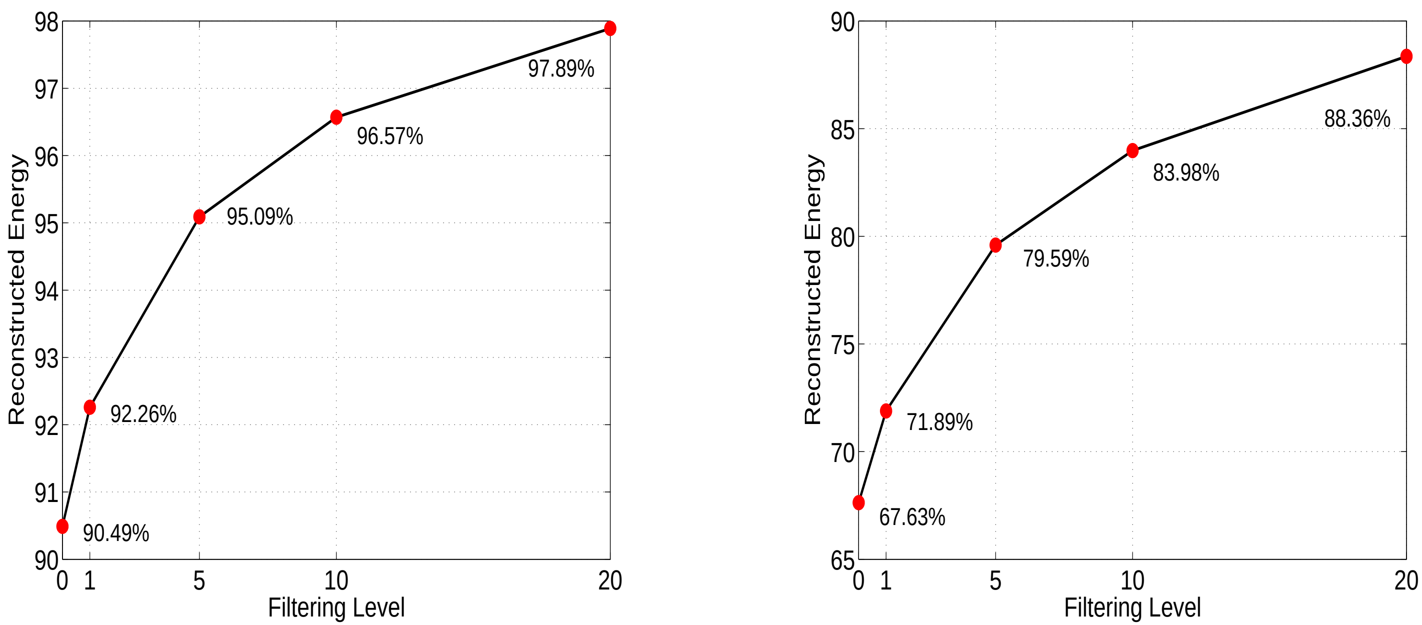
- Major limitation is the ability of the POD modes to adequately represent the flow field.
- Filtering techinque
  - Space average filter:

$$\boldsymbol{u}^*(\boldsymbol{x}_j, t) = \frac{\sum_{p \in I_j} V(C_p) \boldsymbol{u}(\boldsymbol{x}_p, t)}{\sum_{p \in I_j} V(C_p)}$$

where $I_j$ is the ensemble of all the vertex of the neighbouring cells of $C_j$ included itself.

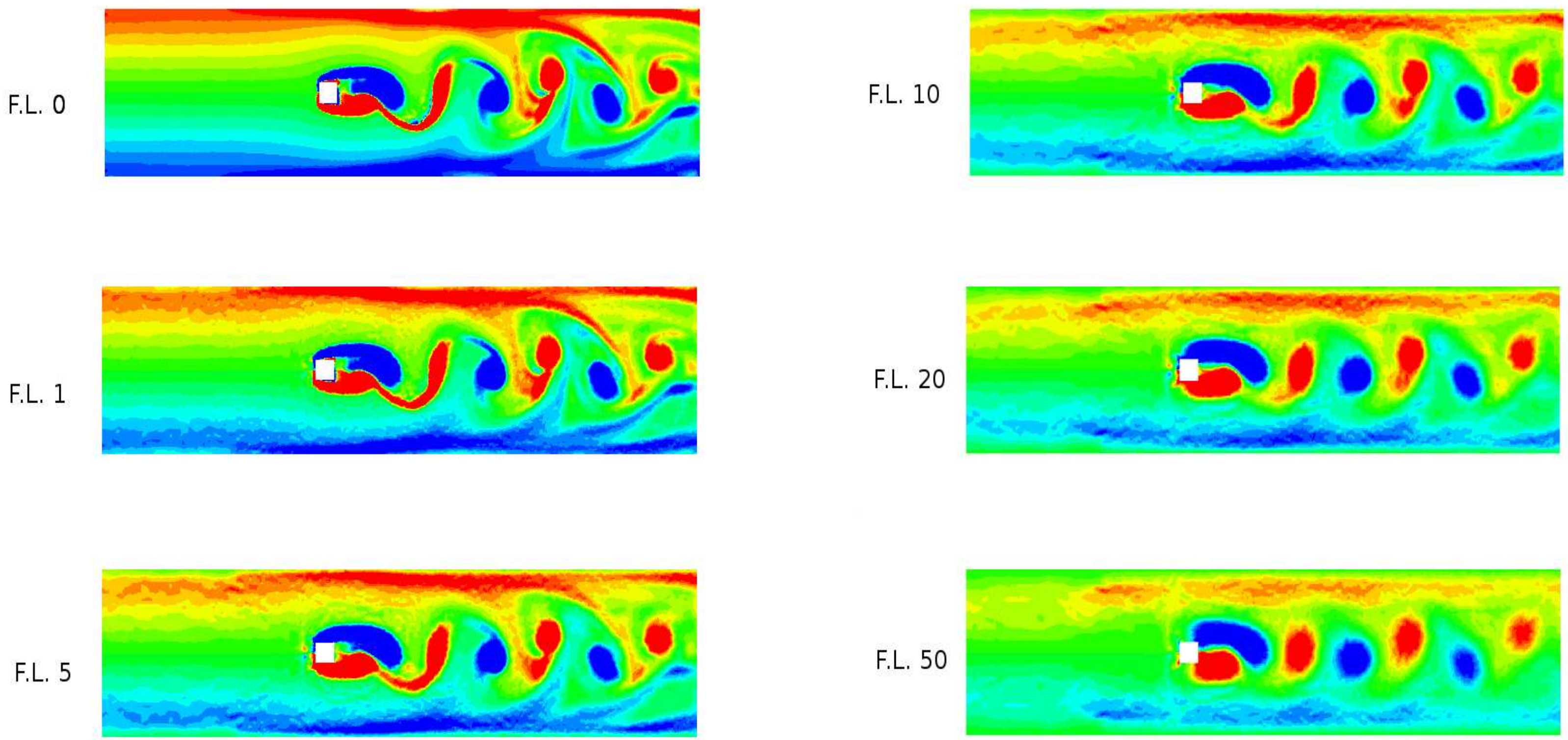
# Filtering Technique

- $N_r = 20$ Space Average Filter - Reconstructed energy inside and outside database

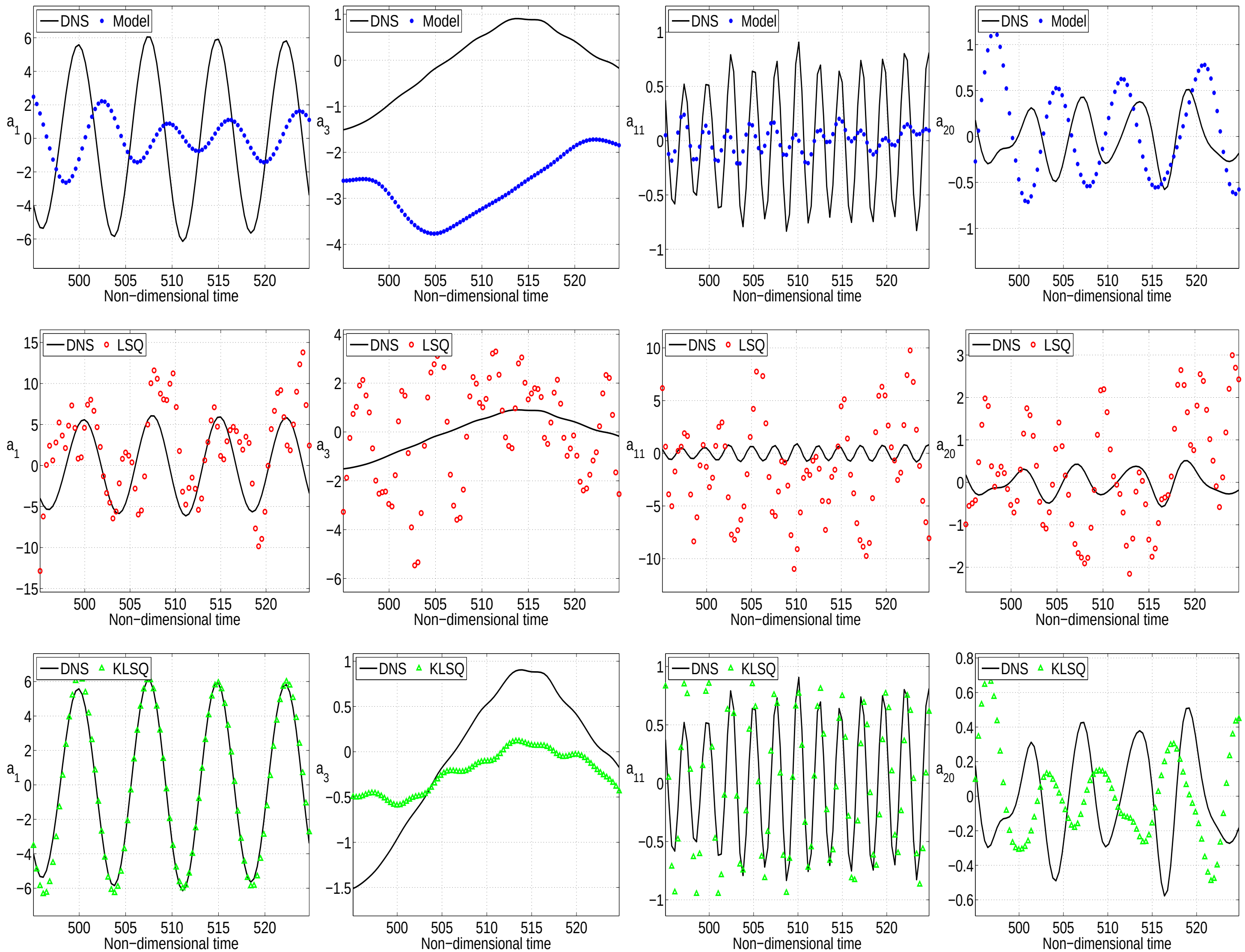


# Filtering Technique

- Space Average Filter



# Results : Modal coefficients prediction




- Some representative modal coefficients estimated vs. DNS projections.
  1st line : POD-ROM ; 2nd line : LSQ ; 3rd line : KLSQ (24 velocity sensors - filtering level 5).

# Reslts : Flow field estimation

| Database | | $\overline{e(U')}$% | $\overline{e(V')}$% | $\overline{e(W')}$% | $\overline{e\ (U)}$% | $\overline{e(V)}$% | $\overline{e(W)}$% |
|---|---|---|---|---|---|---|---|
| No Filt | min | 57.48 | 43.41 | 95.57 | 8.30 | 40.15 | 93.47 |
| | KLSQ | 64.67 | 49.77 | 102.26 | 9.35 | 46.02 | 99.98 |
| Filt 5 | min | 49.41 | 33.56 | 92.37 | 6.39 | 30.91 | 88.77 |
| | KLSQ | 58.57 | 46.23 | 104.27 | 7.58 | 42.57 | 100.27 |
| Filt 10 | min | 46.61 | 29.68 | 90.83 | 5.66 | 27.34 | 86.23 |
| | KLSQ | 54.26 | 40.01 | 104.96 | 6.59 | 36.83 | 99.81 |

- Mean reconstruction error on the U, V, W components for the total and fluctuating field at Re = 300: min is the error using the projection of the DNS velocity fields onto $20$ POD modes.
- **Actual Flow (filtering level 5) vs. Reconstruction (video)**