Improvement of Reduced Order Modeling based on Proper Orthogonal Decomposition

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Summary

Context and flow configuration

I - A pressure extended Reduced Order Model based on POD

II - Stabilization of Reduced Order Models

- ▶ Residuals based stabilization method
- ▶ Classical SUPG and VMS methods

III - Improvement of the functional subspace

- ▶ An hybrid DNS/POD ROM method (Database modification)
- ▶ Krylov like method

Conclusions

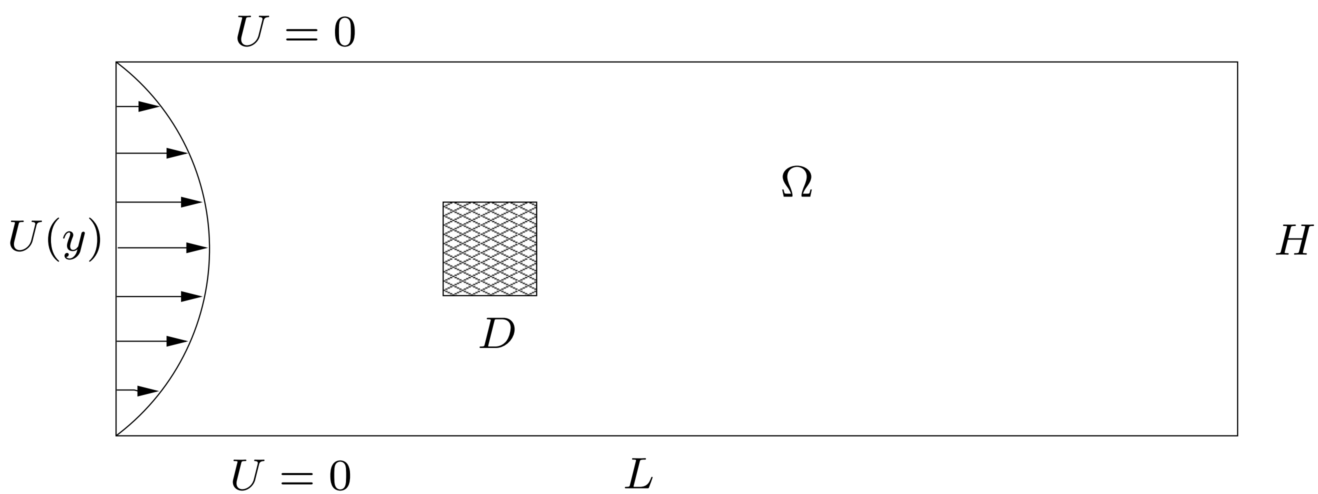
Context and flow configuration

▷ **Context**

- Need of Reduced Order Model for Flow Control Purpose
  - ↪ To reduce the CPU time
  - ↪ To reduce the memory storage during adjoint-based minimization process
- Optimization + POD ROM methods
  - ↪ Generalized basis, no POD basis actualization : fast but no "convergence" proof
  - ↪ Trust Region POD (TRPOD), POD basis actualization : proof of convergence !
- Drawbacks
  - ↪ Need to stabilize POD ROM (lack of dissipation, roundoff errors, pressure term)
  - ↪ Basis actualization : DNS → high numerical costs !
- Solutions
  - ↪ Efficient ROM & stabilization
  - ↪ Low costs functional subspace adaptation during optimization process

Context and flow configuration

- **Flow Configuration**
  - 2-D Confined flow past a square cylinder in laminar regime
  - Viscous fluid, incompressible and newtonian
  - No control



- **Numerical methods**
  - Penalization method for the square cylinder
  - Multigrids V-cycles method in space
  - Gear method in time

C.-H. Bruneau solver

I - A pressure extended Reduced Order Model

▶ **Momentum conservation**

Detailled model (exact)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \frac{1}{Re}\Delta \boldsymbol{u}$$

Temporal discretization

$$\frac{\boldsymbol{u}^{n+1}}{\Delta t} + \boldsymbol{\nabla} p^{n+1} - \frac{1}{Re}\Delta \boldsymbol{u}^{n+1} = \frac{\boldsymbol{u}^{n}}{\Delta t} - (\boldsymbol{u}^{n} \cdot \boldsymbol{\nabla})\boldsymbol{u}^{n}$$

Projection onto the pressure extended POD basis (correlations onto $\boldsymbol{U} = (\boldsymbol{u},\, p)^T$)

$$\widetilde{\boldsymbol{u}}(\boldsymbol{x},\, t) = \sum_{i=1}^{N} a_i(t)\boldsymbol{\phi}_i(\boldsymbol{x}) \;\text{ and }\; \widetilde{p}(\boldsymbol{x},\, t) = \sum_{i=1}^{N} a_i(t)\psi_i(\boldsymbol{x})$$

$$\sum_{j=1}^{N} a_j^{(n+1)} \left(\frac{\boldsymbol{\phi}_j}{\Delta t} + \boldsymbol{\nabla}\psi_j - \frac{1}{Re}\Delta \boldsymbol{\phi}_j \right) = \sum_{i=j}^{N} a_j^{(n)} \frac{\boldsymbol{\phi}_j}{\Delta t} + \left(\sum_{j=1}^{N} a_j^{(n)} \boldsymbol{\phi}_j^{(\boldsymbol{u})} \cdot \boldsymbol{\nabla} \right) \sum_{k=1}^{N} a_k^{(n)} \boldsymbol{\phi}_k^{(\boldsymbol{u})}$$

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I - A pressure extended Reduced Order Model

After some simplifications

$$\sum_{j=1}^{N} a_j^{(n+1)} \left(\frac{\boldsymbol{\phi}_i}{\Delta t} + \boldsymbol{\nabla}\psi_i - \frac{1}{Re}\Delta\boldsymbol{\phi}_i \right) = \sum_{j=1}^{N} a_j^{(n)} \frac{\boldsymbol{\phi}_j}{\Delta t} + \sum_{j=1}^{N}\sum_{k=1}^{N} a_j^{(n)} \left(\boldsymbol{\phi}_j^{(\boldsymbol{u})} \cdot \boldsymbol{\nabla} \right) \boldsymbol{\phi}_k^{(\boldsymbol{u})} a_k^{(n)}$$

$$\sum_{j=1}^{N} a_j^{(n+1)} \boldsymbol{\chi}_j = \sum_{j=1}^{N} a_j^{(n)} \boldsymbol{\xi}_j + \sum_{j=1}^{N}\sum_{k=1}^{N} a_j^{(n)} \boldsymbol{\zeta}_{jk} a_k^{(n)}$$

Least squares

$$\sum_{j=1}^{N} \boldsymbol{\chi}_i^T \boldsymbol{\chi}_j a_j^{(n+1)} = \sum_{j=1}^{N} \boldsymbol{\chi}_i^T \boldsymbol{\xi}_j a_j^{(n)} + \sum_{j=1}^{N}\sum_{k=1}^{N} a_j^{(n)} \boldsymbol{\chi}_i^T \boldsymbol{\zeta}_{jk} a_k^{(n)}$$

$$\sum_{j=1}^{N} L_{ij}^{qdm} a_j^{(n+1)} = \sum_{j=1}^{N} B_{ij} a_j^{(n)} + \sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk} a_j^{(n)} a_k^{(n)}$$

$\hookrightarrow$ The ROM does not satisfied *a priori* the mass conservation
(for non divergence free modes, as NSE-Residual modes)

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I - A pressure extended Reduced Order Model

▶ **Mass conservation**

Detailled model

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

Projection onto the POD basis

$$\sum_{j=1}^{N} a_j^{(n+1)} \boldsymbol{\nabla} \cdot \boldsymbol{\phi}_j = \mathbf{0}$$

$$\sum_{j=1}^{N} (\boldsymbol{\nabla} \cdot \boldsymbol{\phi}_j)^T \boldsymbol{\nabla} \cdot \boldsymbol{\phi}_j a_j^{(n+1)} = \mathbf{0}$$

$$\sum_{j=1}^{N} L_{ij}^{div} a_j^{(n+1)} = \mathbf{0}$$

Modified ROM

$$\sum_{j=1}^{N} (\alpha L_{ij}^{qdm} + \beta L_{ij}^{div}) a_j^{(n+1)} = \sum_{j=1}^{N} \alpha B_{ij} a_j^{(n)} + \sum_{j=1}^{N} \sum_{k=1}^{N} \alpha C_{ijk} a_j^{(n)} a_k^{(n)}$$

$\hookrightarrow$ The ROM has moreover to satisfy the flow rate conservation..

I - A pressure extended Reduced Order Model

▶ **Flow rate conservation**

For the 2-D confined flow

$$\int_{\mathcal{S}} u \, d\mathcal{S} = Cste$$

$$\sum_{i=1}^{N} a_j(t) \int_{\mathcal{S}} \boldsymbol{\phi}_j^u \, d\mathcal{S} = Cste$$

$$\sum_{j=1}^{N} \frac{da_j}{dt} \int_{\mathcal{S}} \boldsymbol{\phi}_j^u \, d\mathcal{S} = 0$$

$$\sum_{j=1}^{N} L_{ij}^{deb} a_j^{(n+1)} = \sum_{j=1}^{N} L_{ij}^{deb} a_j^{(n)}$$

Finally, the ROM writes

$$\boxed{\sum_{j=1}^{N} (\alpha L_{ij}^{qdm} + \beta L_{ij}^{div} + \gamma L_{ij}^{deb}) a_j^{(n+1)} = \sum_{j=1}^{N} (\alpha B_{ij} + \gamma L_{ij}^{deb}) a_j^{(n)} + \sum_{j=1}^{N} \sum_{k=1}^{N} \alpha C_{ijk} a_j^{(n)} a_k^{(n)}}$$

I - A pressure extended Reduced Order Model

- **Advantage** no modelisation of the pressure term

$Re = 200$, 11 modes $\Rightarrow$ convergence towards the exact limit cycles ($=$ DNS)

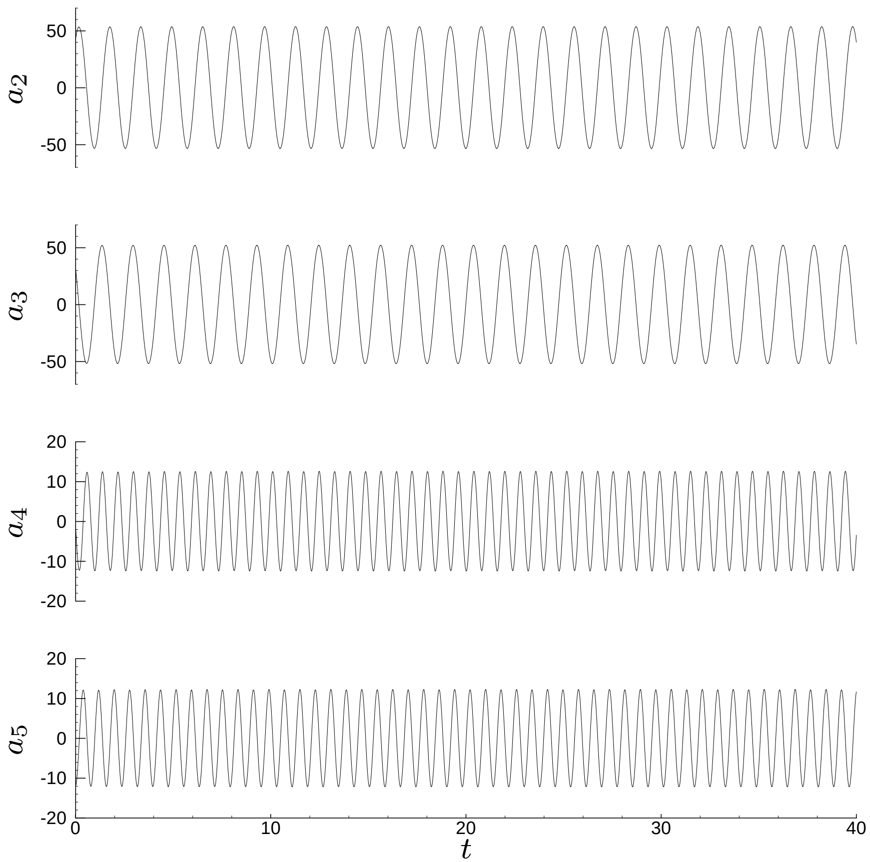


Fig. : *Temporal evolution of the POD ROM coefficients over 25 vortex shedding periods*

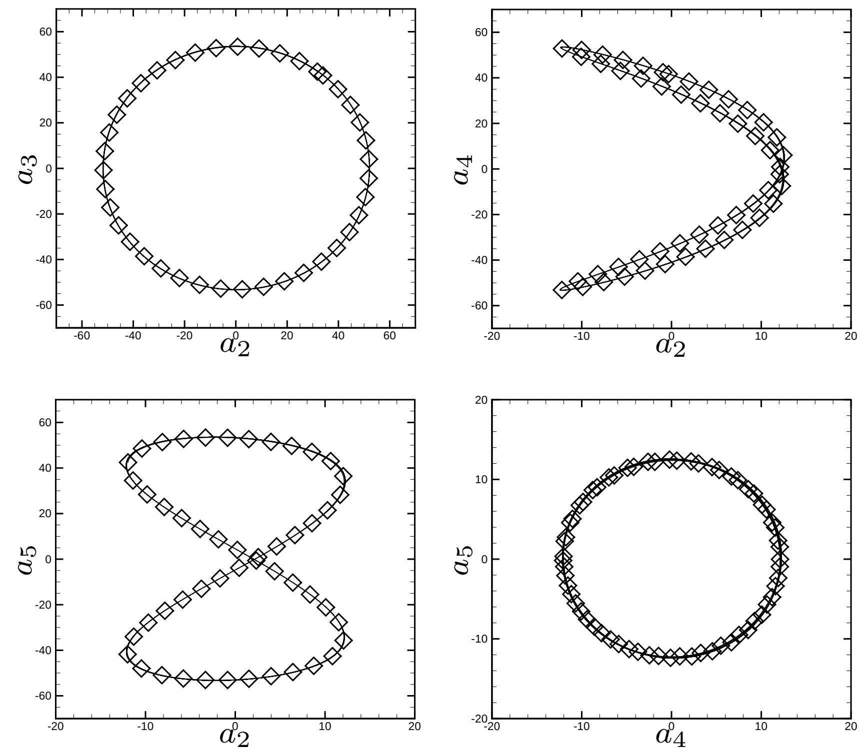
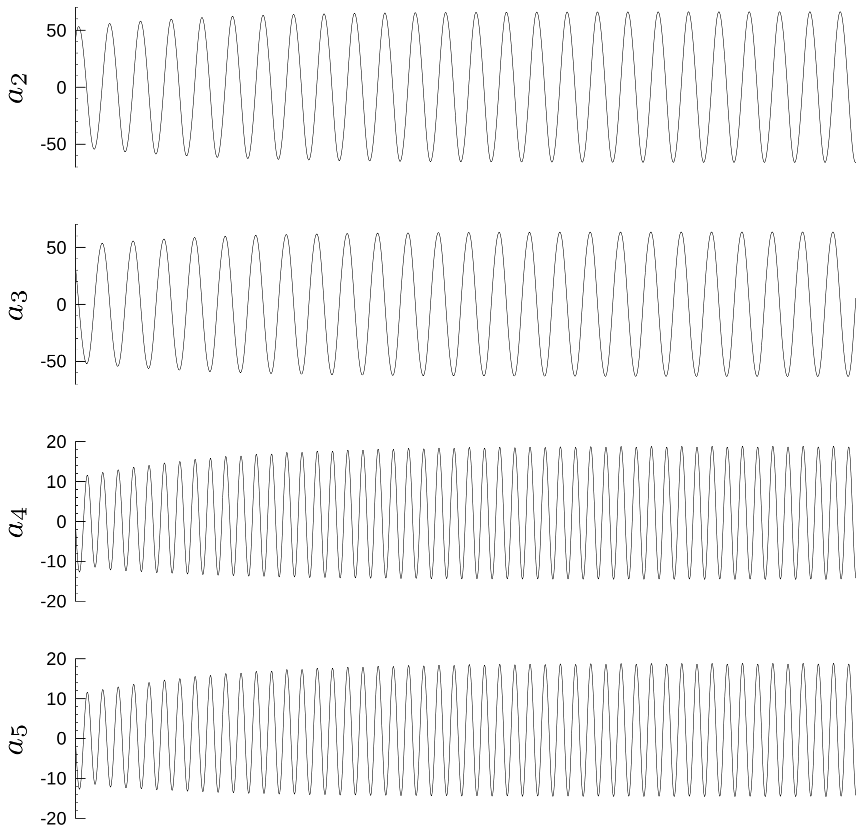


Fig. : *Limit cycles of the POD ROM coefficients over 25 vortex shedding periods*

I - A pressure extended Reduced Order Model

▶ **Drawbaks** same as usual, *i.e.* lack of dissipation...

$Re = 200$, 5 modes $\Rightarrow$ convergence towards an erroneous limit cycles ($\neq$ DNS)



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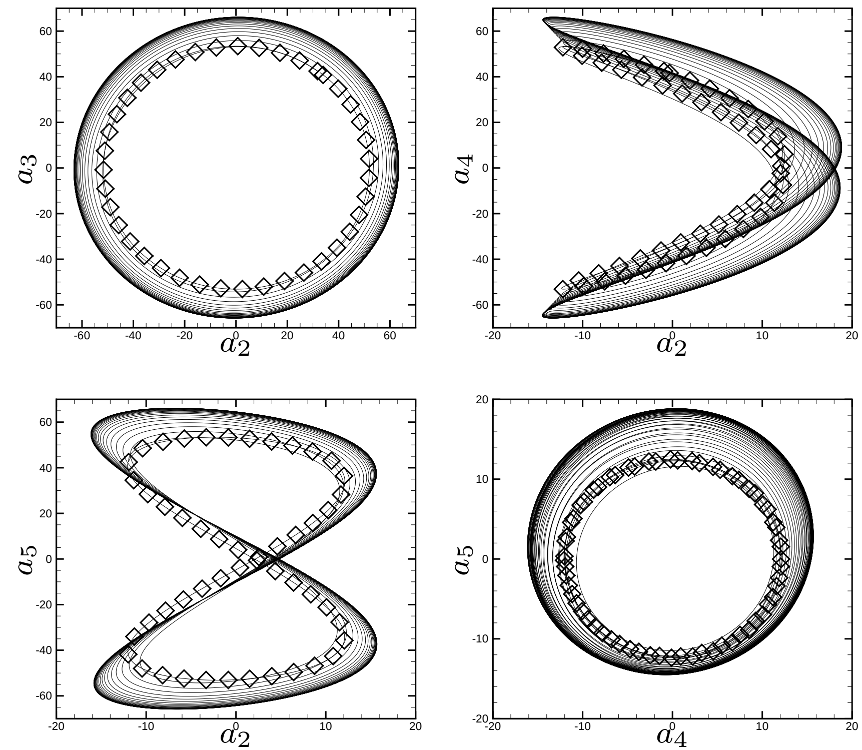

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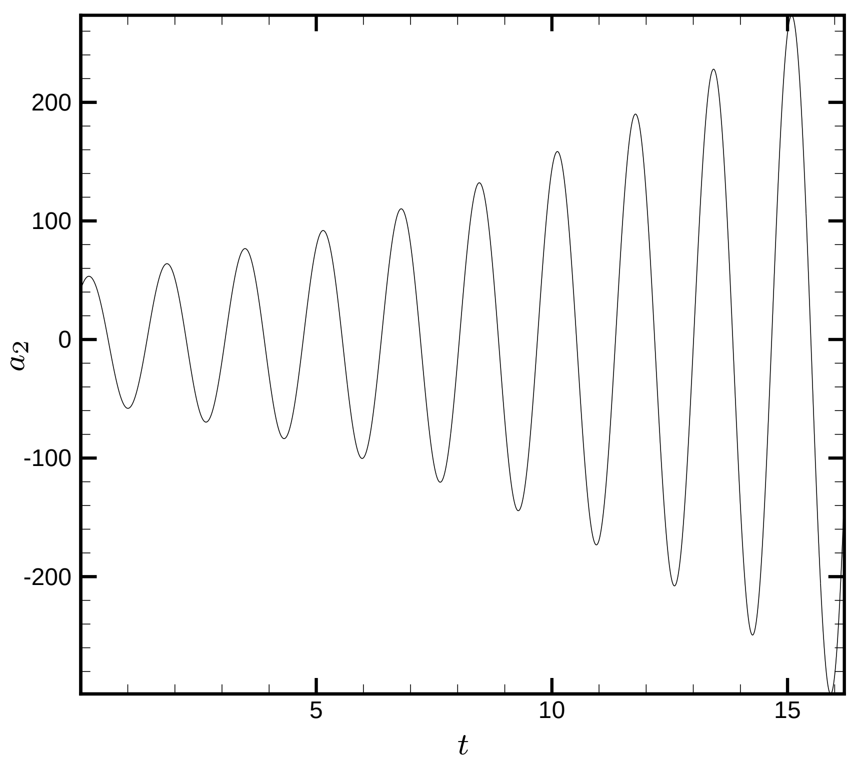
I - A pressure extended Reduced Order Model

▶ **Drawbaks** same as usual, *i.e.* lack of dissipation...

$Re = 200$, 3 modes $\Rightarrow$ exponential divergence



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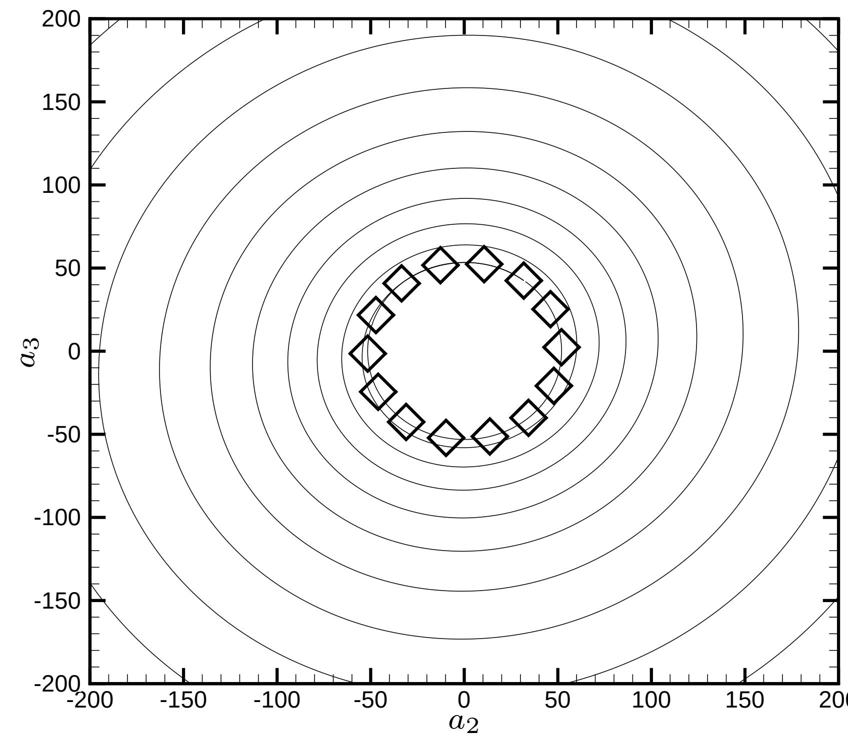

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II - POD ROM stabilization

▶ **Overview of stabilization methods (non-exhaustive)**

- Eddy viscosity
  - ↪ Heisenberg viscosity
  - ↪ Spectral vanishing viscosity
  - ↪ Optimal viscosity
- Penalty method
- Calibration of POD ROM coefficients

▶ **"New" stabilization methods in POD ROM context**

- *Residuals based stabilization method*
- *Streamline Upwind Petrov-Galerkin (SUPG) and Variational Multi-scale (VMS) methods*

II - POD ROM stabilization

▶ **Residuals based stabilization method**

$\Rightarrow$ **Idea** add dominant POD-NSE residual modes to the existing basis

$\hookrightarrow$ The POD-NSE residuals are $\mathcal{L}(\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t), \widetilde{p}(\boldsymbol{x}, t)) = \boldsymbol{R}(\boldsymbol{x}, t)$,
where $\widetilde{\boldsymbol{u}}$ and $\widetilde{p}$ obtained using POD and $\mathcal{L}$ is the NSE operator

- **Model $A^{[N]}$**, **unstable POD ROM** built with N basis functions $\boldsymbol{\Phi}_i(\boldsymbol{x})$.

Algorithm

1. Integrate the ROM to obtain $a_i(t)$ and extract N_s snapshots $a_i(t_k)$, $k = 1, \ldots, N_s$.
2. Compute $\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t_k) = \sum_{i=1}^{N} a_i(t_k)\boldsymbol{\phi}_i(\boldsymbol{x})$, $\widetilde{p}(\boldsymbol{x}, t_k) = \sum_{i=1}^{N} a_i(t_k)\psi_i(\boldsymbol{x})$, and $\boldsymbol{R}(\boldsymbol{x}, t_k)$.
3. Compute the POD modes $\boldsymbol{\Psi}(\boldsymbol{x})$ of the NSE residuals.
4. Add the K firsts residual modes $\boldsymbol{\Psi}(\boldsymbol{x})$ to the existing POD basis $\boldsymbol{\Phi}_i(\boldsymbol{x})$ and built a new ROM (here the mass and flow rate constraints are important).

- **Model $B^{[N;K]}$**, **PODRES ROM** built with N POD basis functions $\boldsymbol{\Phi}_i(\boldsymbol{x})$
$+ K$ RES basis functions $\boldsymbol{\Psi}_i(\boldsymbol{x})$

II - POD ROM stabilization

▶ **SUPG and VMS methods**

⇒ **Idea** approximate the fine scales using the NSE residuals $\boldsymbol{R} = (\boldsymbol{R}_M, R_C)^T$

$$\boldsymbol{u}'(\boldsymbol{x}, t) = \tau_M \boldsymbol{R}_M(\boldsymbol{x}, t) \quad \text{and} \quad p'(\boldsymbol{x}, t) = \tau_C R_C(\boldsymbol{x}, t)$$

↪ Class of penalty methods, *i.e.*

$$\sum_{j=1}^{N} L_{ij} \frac{da_j}{dt} = \sum_{j=1}^{N} B_{ij} a_j + \sum_{j=1}^{N} \sum_{k=1}^{N} C_{ijk} a_j a_k + F_i(t)$$

- **Model $C^{[N]}$, SUPG method**

$$F_i^{SUPG}(t) = (\widetilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{\Phi}_i + \nabla \Psi_i, \tau_M \boldsymbol{R}_M(\boldsymbol{x}, t))_\Omega + (\nabla \cdot \boldsymbol{\Phi}_i, \tau_C R_C(\boldsymbol{x}, t))_\Omega$$

- **Model $D^{[N]}$, VMS method**

$$F_i^{VMS}(t) = F_i^{SUPG}(t) + (\widetilde{\boldsymbol{u}} \cdot (\nabla \boldsymbol{\Phi}_i)^T, \tau_M \boldsymbol{R}_M(\boldsymbol{x}, t))_\Omega - (\nabla \boldsymbol{\Phi}_i, \tau_M \boldsymbol{R}_M(\boldsymbol{x}, t) \otimes \tau_M \boldsymbol{R}_M(\boldsymbol{x}, t))_\Omega$$

↪ Parameters τ_M and τ_C are determined using adjoint based minimization method

II - POD ROM stabilization

- $Re = 200$ and $N = 5$ POD basis function $\rightarrow$ erroneous limit cylcles

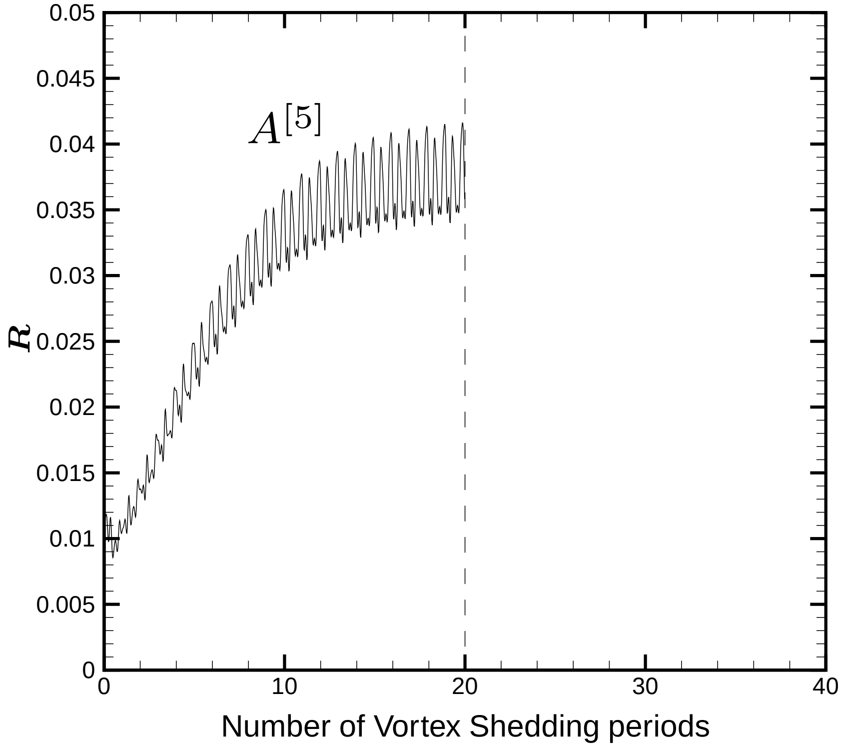
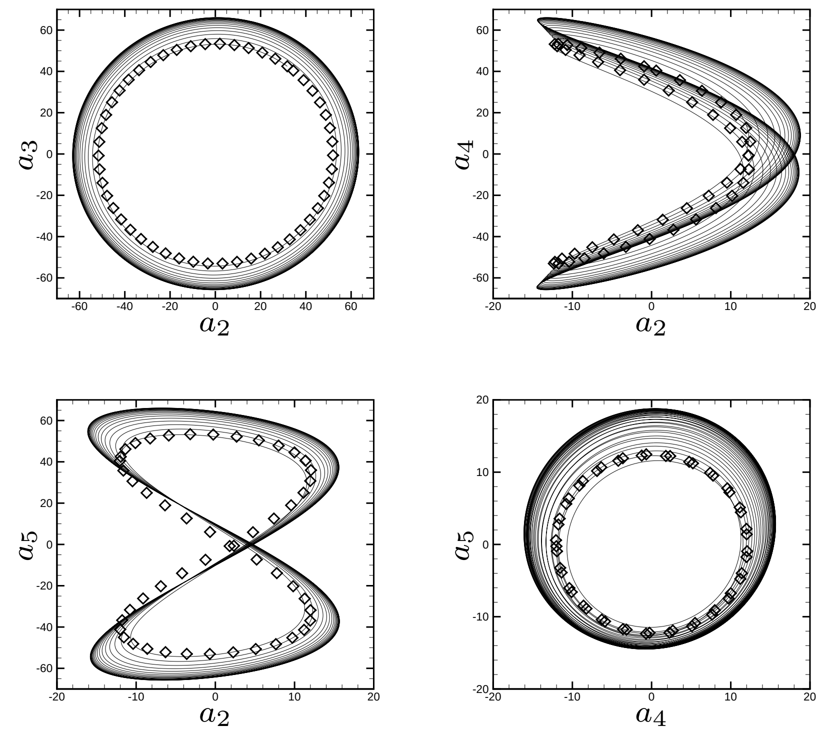


Fig. : *temporal evolution of the L_2 norm of the POD-NSE residuals*



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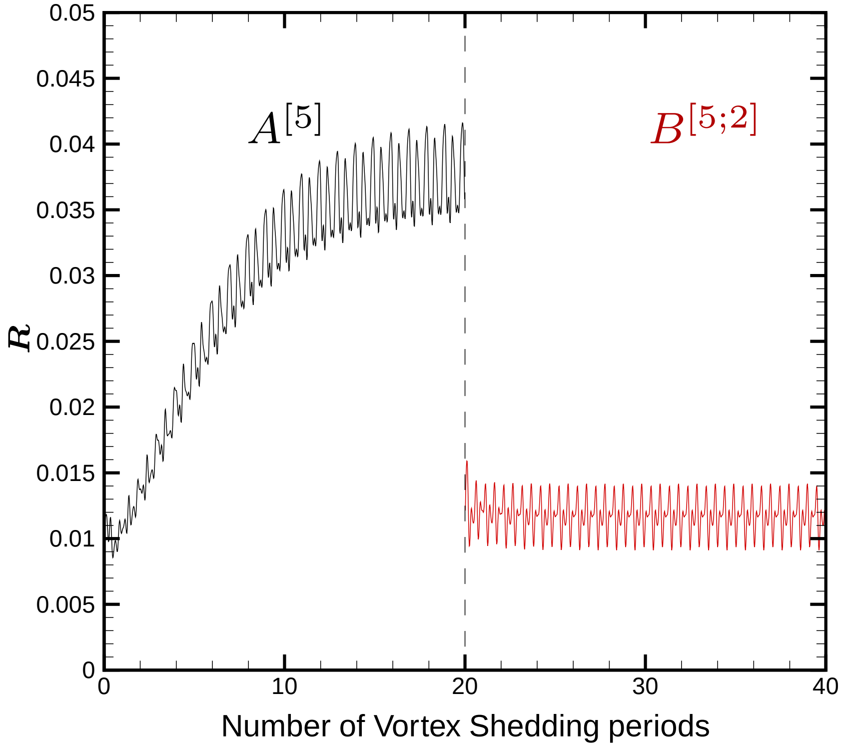

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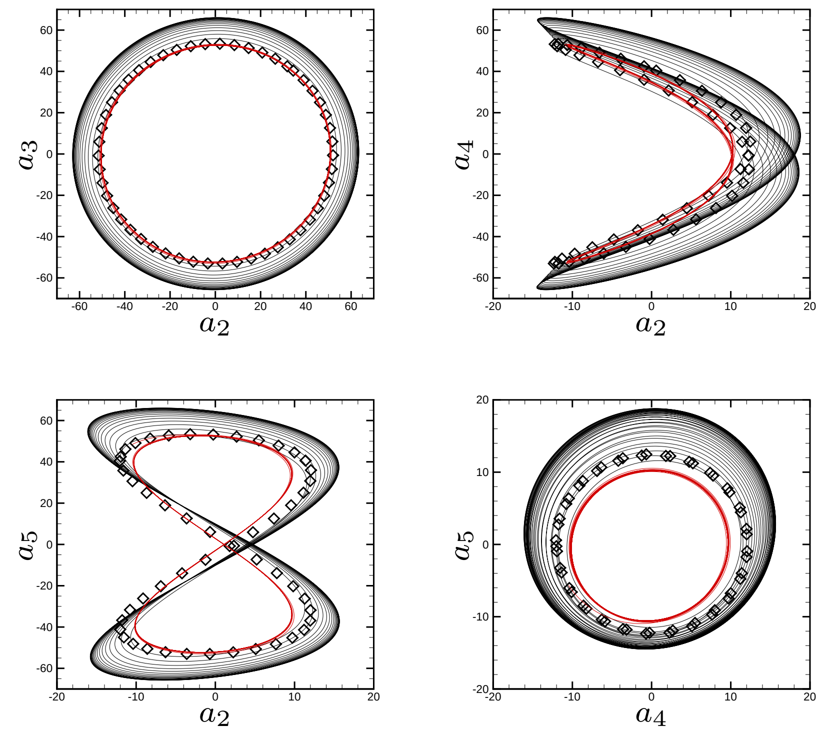

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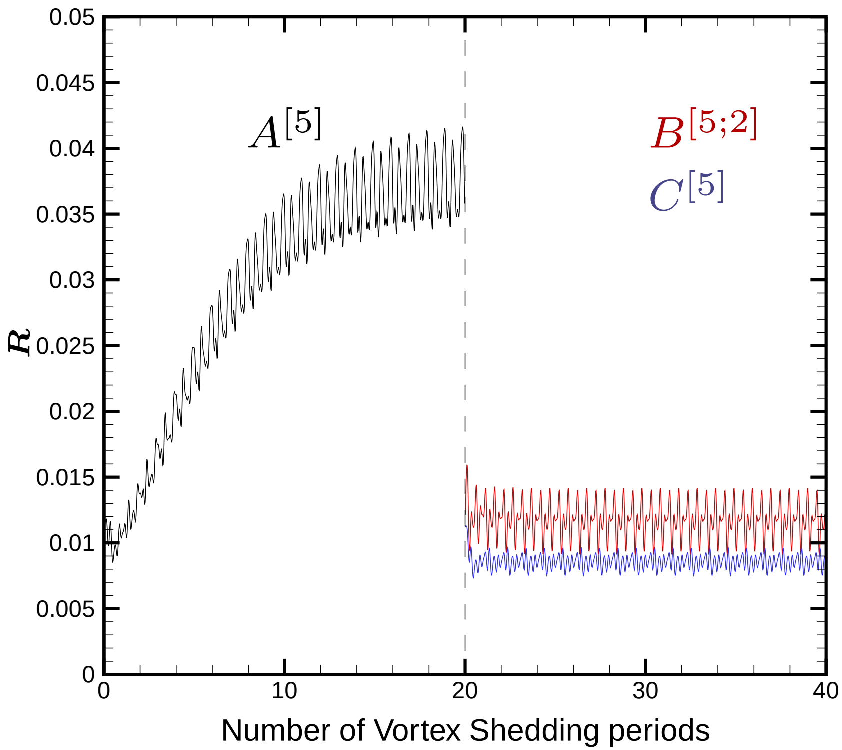

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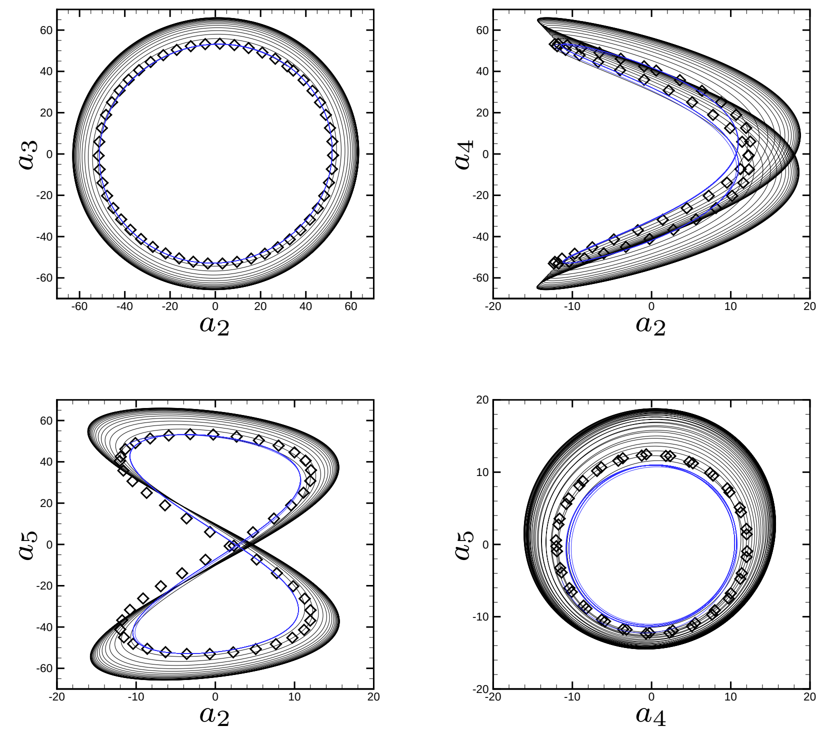

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II - POD ROM stabilization

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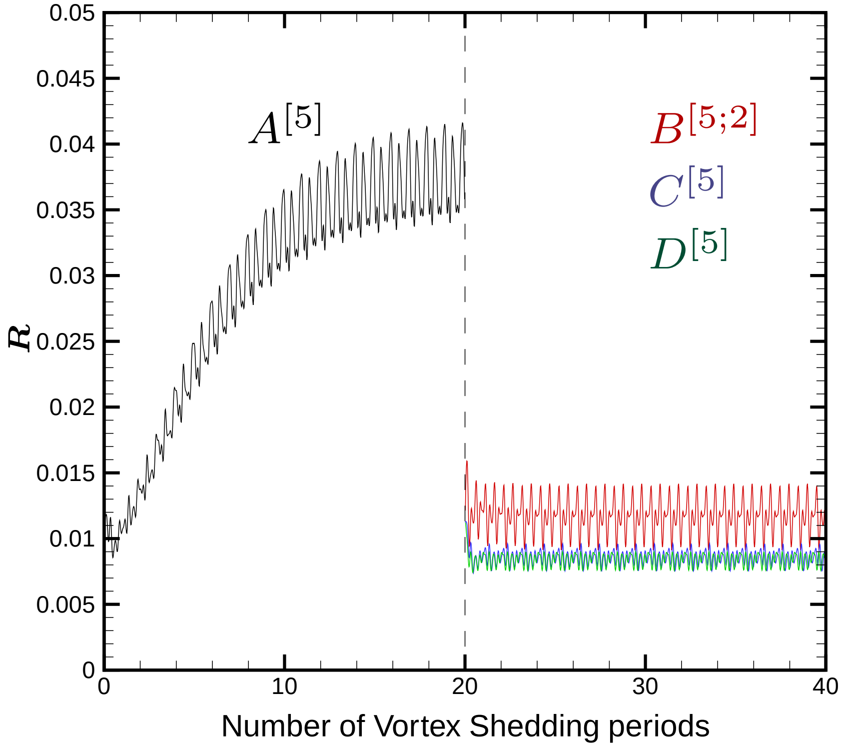

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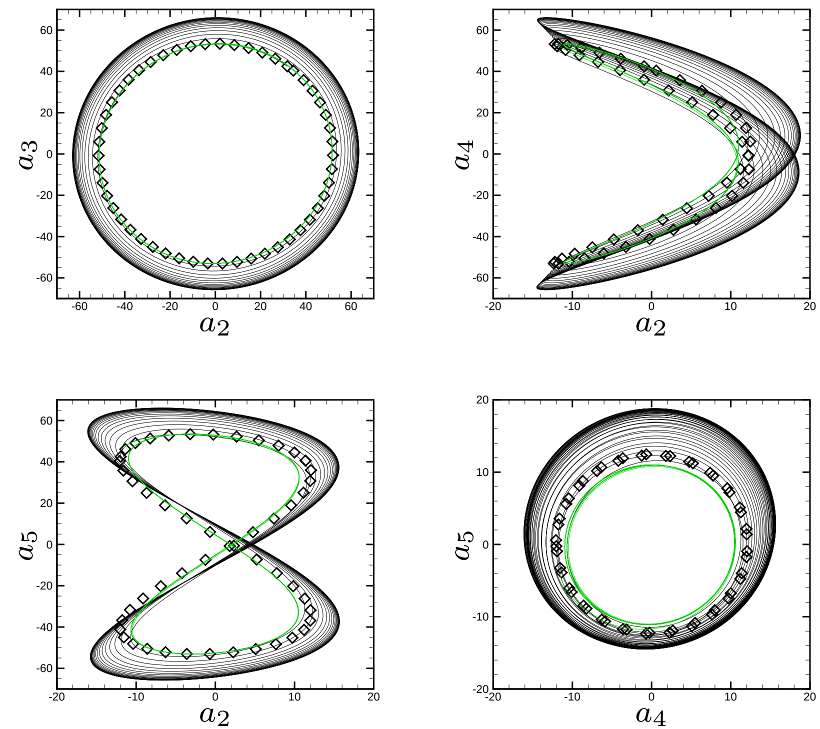

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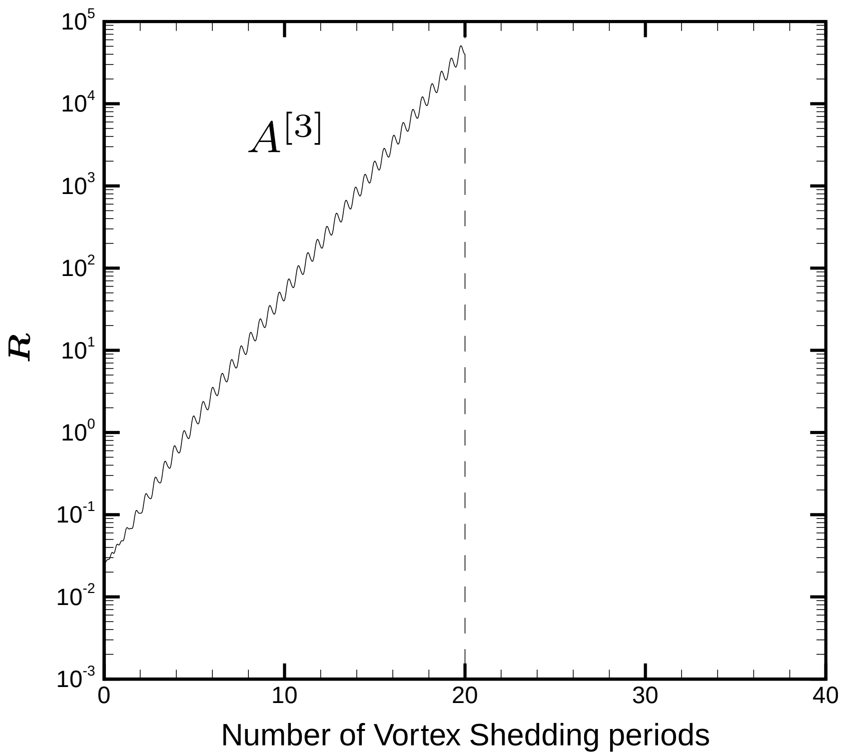
II - POD ROM stabilization

- **$Re = 200$ and $N = 3$ POD basis function $\rightarrow$ divergence**



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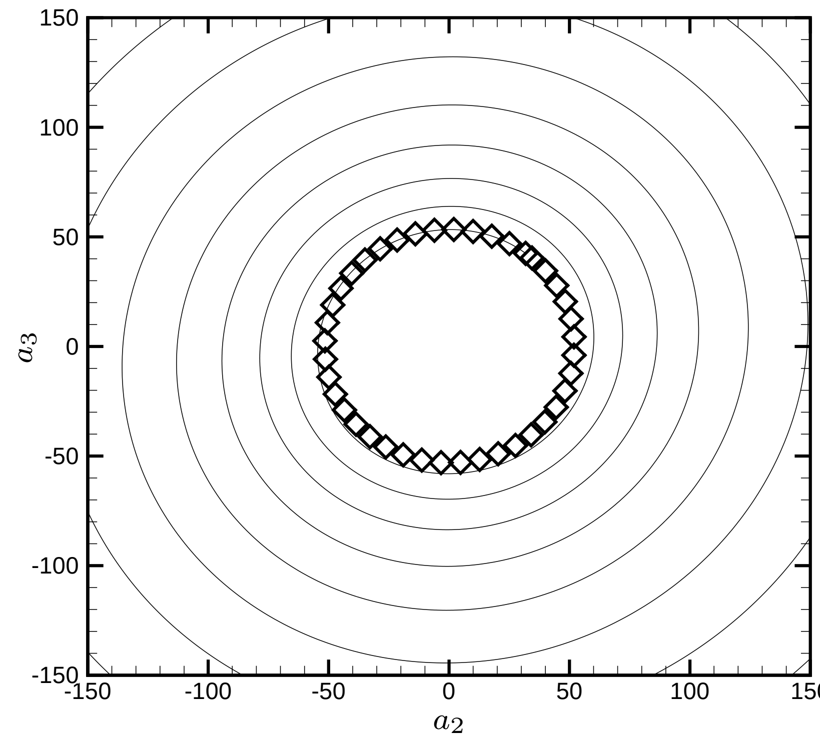

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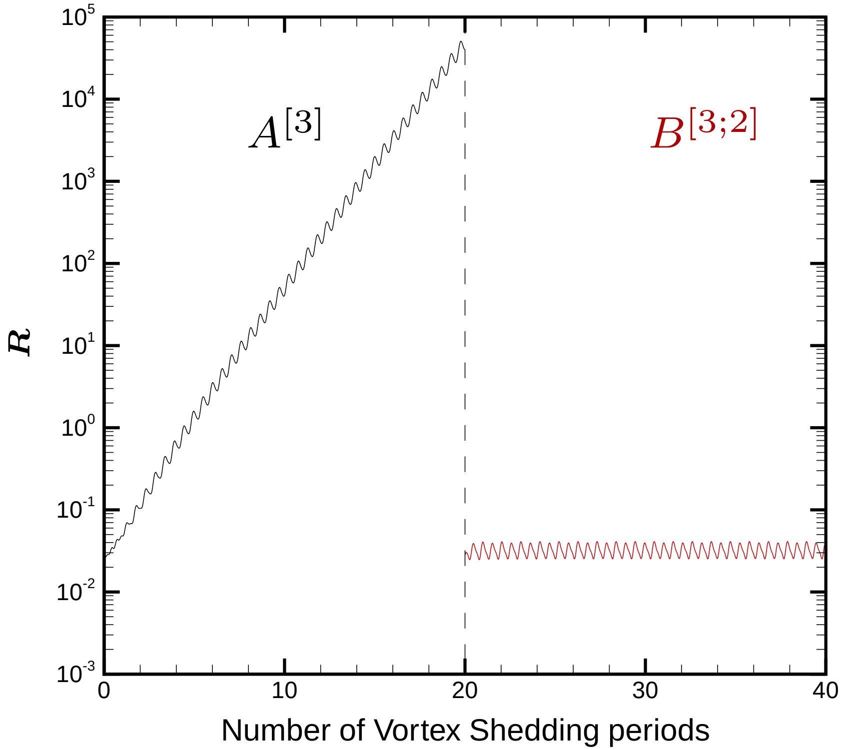
II - POD ROM stabilization

- $Re = 200$ **and** $N = 3$ **POD basis function** $\rightarrow$ **divergence**



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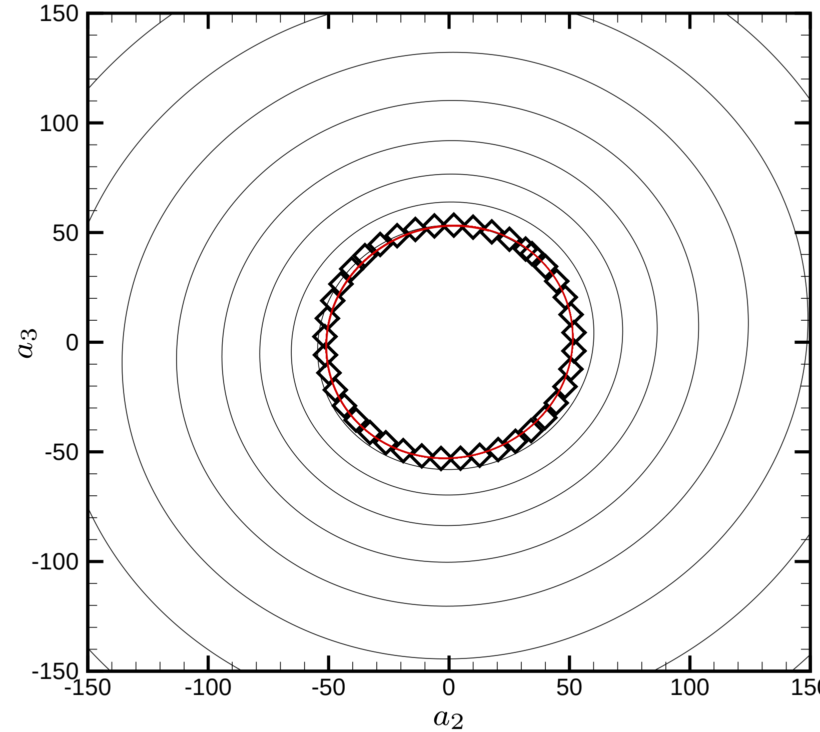

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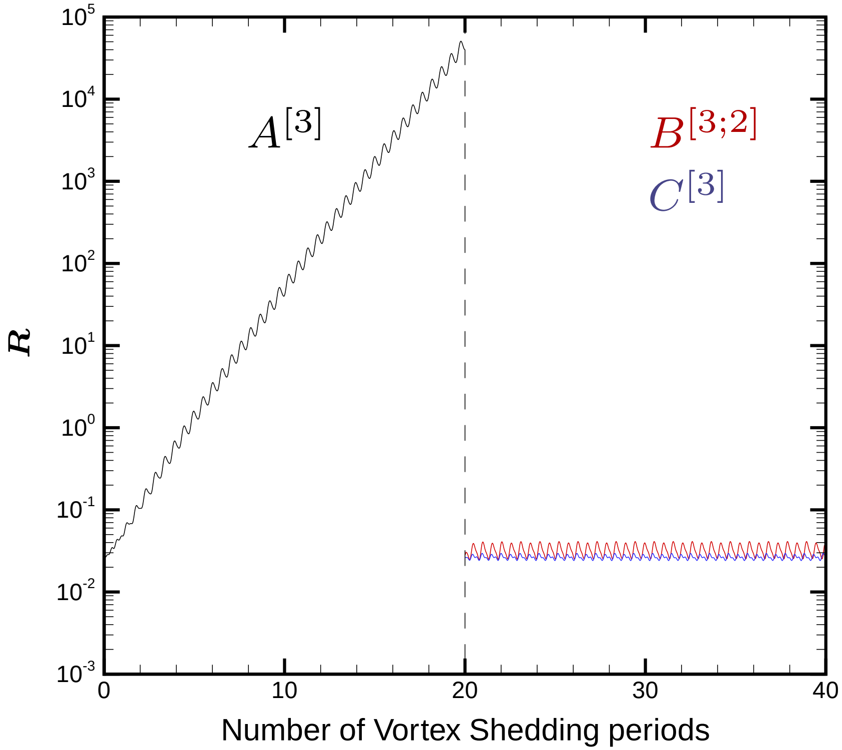

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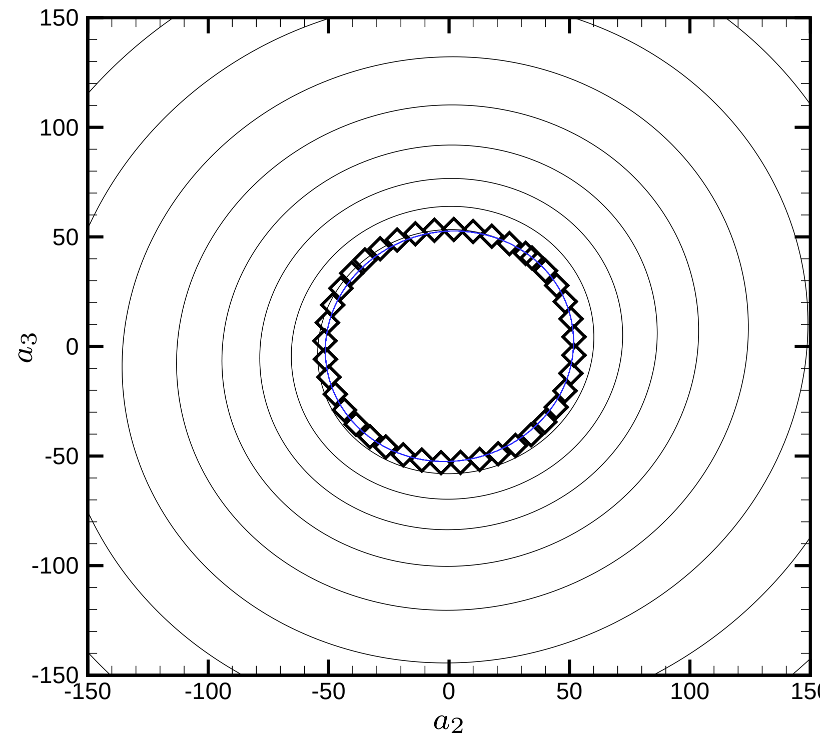

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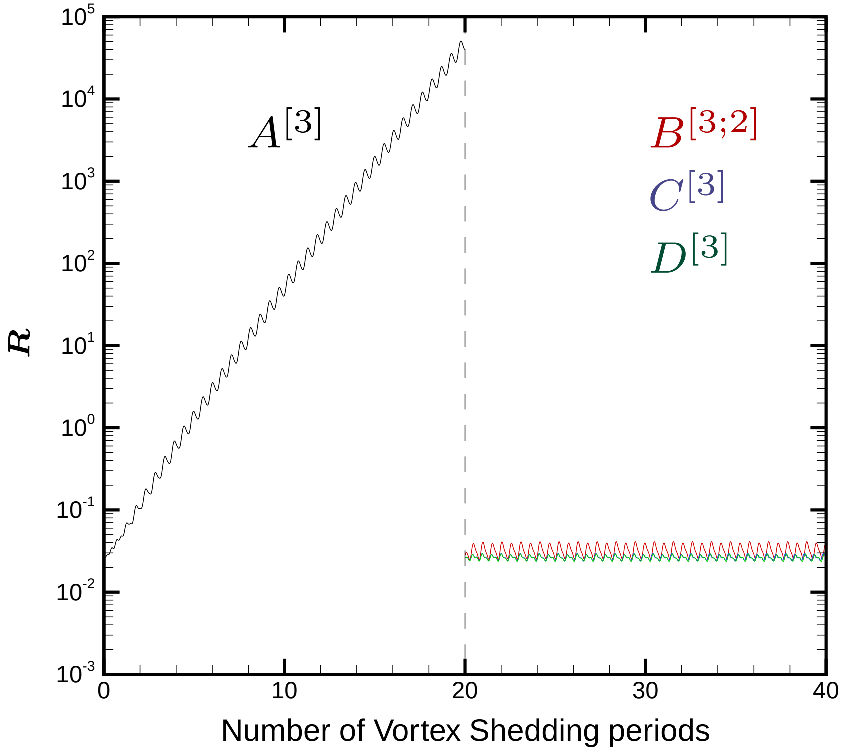

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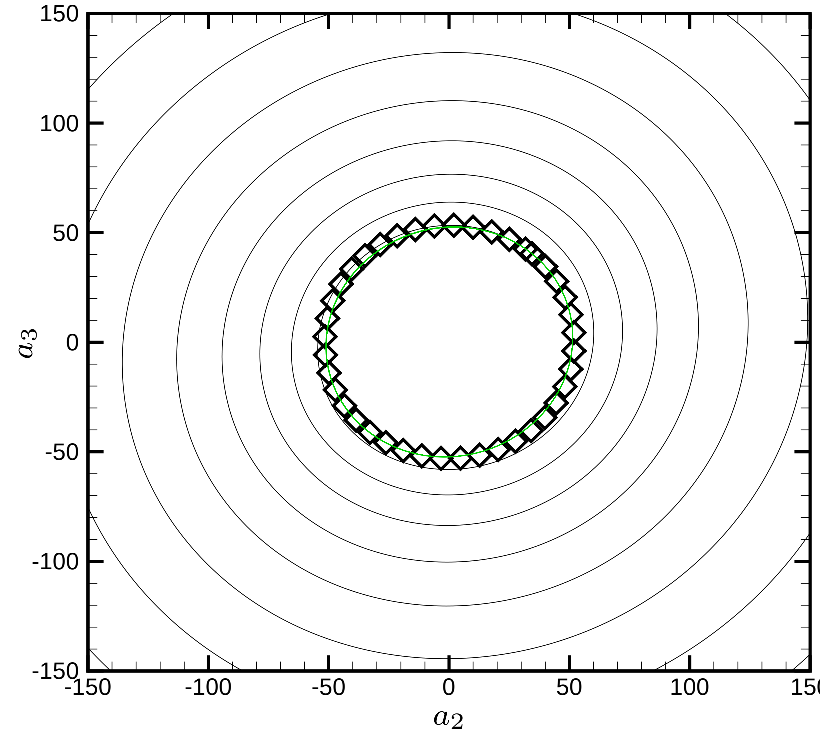

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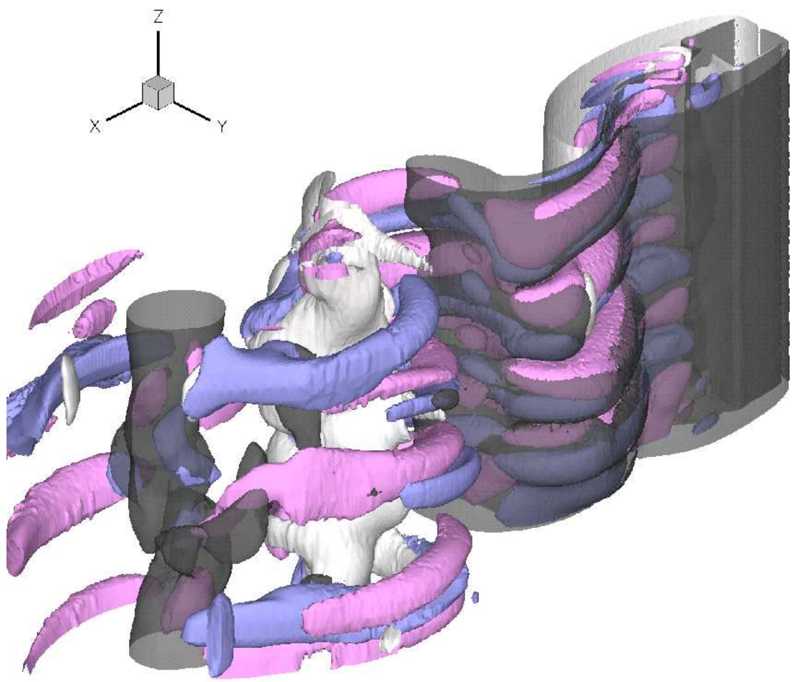
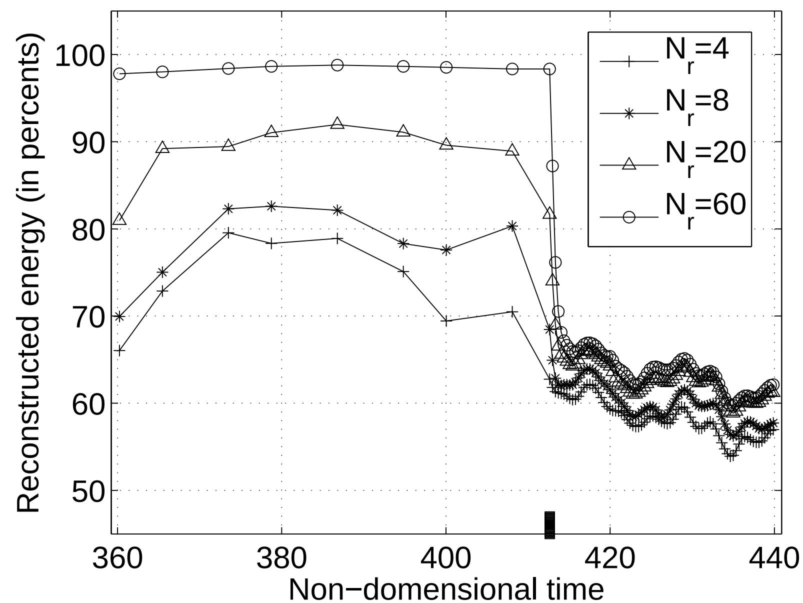
III - Improvement of the functional subspace

▶ **Functional subspace drawbacks**, $\boldsymbol{\Phi}_n(\boldsymbol{x})$: lack of representativity of 3D flows *outside the database*





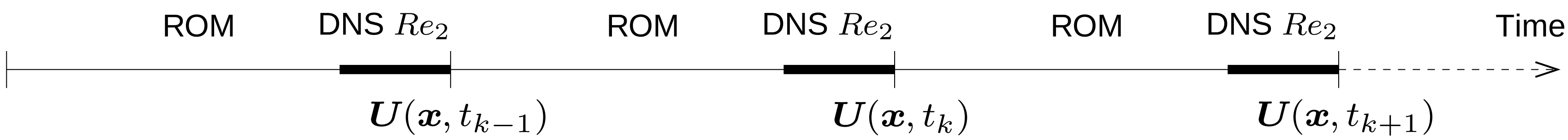
Figures results from Buffoni *etal.* Journal of Fluid Mech. **569** (2006)

- Problems for 3D flow control
- Erroneous turbulence properties (spectrum, *etc*)

III - Improvement of the functional subspace

▷ **Method 1 : hybrid ROM-DNS method to adapt the functional subspace $\boldsymbol{\Phi}_n(\boldsymbol{x})$**
Goal : determine $\boldsymbol{\Phi}_n(\boldsymbol{x})$ at Re_2 starting from $\boldsymbol{\Phi}_n(\boldsymbol{x})$ at Re_1 for low numerical costs.

○ ***Database modification :*** statistics evolution $\Rightarrow \varphi : \Phi^{(k)} \mapsto \Phi^{(k+1)}$



1. Database modification $[\boldsymbol{U}(\boldsymbol{x},\, t_1)\; \boldsymbol{U}(\boldsymbol{x},\, t_2)\; \ldots \boldsymbol{U}(\boldsymbol{x},\, t_{N_r})\,]$

$$\widetilde{\boldsymbol{U}}^{[1,\cdots,N_r]}(\boldsymbol{x}, t_k) = \sum_{n=1}^{N_r} a_n(t_k)\boldsymbol{\phi}_n(\boldsymbol{x}),$$

One snapshot modification using few DNS iterations

$$\boldsymbol{U}(\boldsymbol{x}, t_s) = \widetilde{\boldsymbol{U}}^{[1,\cdots,N_r]}(\boldsymbol{x}, t_s) + \boldsymbol{U}_s^{\perp}(\boldsymbol{x}, t_s).$$

In a general way

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}, t_k) = \widetilde{\boldsymbol{U}}^{[1,\cdots,N_r]}(\boldsymbol{x}, t_k) + \delta_{ks}\boldsymbol{U}^{\perp}(\boldsymbol{x}, t_s),$$

III - Improvement of the functional subspace

2 Modification temporal correlations tensor

$$C(t_k,t_l) = (\boldsymbol{U}(\boldsymbol{x},t_k),\boldsymbol{U}(\boldsymbol{x},t_l))_\Omega$$

$$= \left(\sum_{i=1}^{N_r} a_i(t_k)\boldsymbol{\phi}_i(\boldsymbol{x}) + \boldsymbol{U}^\perp(\boldsymbol{x},t_k), \sum_{j=1}^{N_r} a_j(t_l)\boldsymbol{\phi}_j(\boldsymbol{x}) + \boldsymbol{U}^\perp(\boldsymbol{x},t_l) \right)_\Omega$$

$$= \sum_{i=1}^{N_r}\sum_{j=1}^{N_r} a_i(t_k)a_j(t_l) \underbrace{(\boldsymbol{\phi}_i(\boldsymbol{x}),\boldsymbol{\phi}_j(\boldsymbol{x}))_\Omega}_{=\delta_{ij}} + \left(\boldsymbol{U}^\perp(\boldsymbol{x},t_k),\boldsymbol{U}^\perp(\boldsymbol{x},t_l)\right)_\Omega$$

$$+ \sum_{i=1}^{N_r} a_i(t_k) \underbrace{\left(\boldsymbol{\phi}_i(\boldsymbol{x}),\boldsymbol{U}^\perp(\boldsymbol{x},t_l)\right)_\Omega}_{=0} + \sum_{j=1}^{N_r} a_{lj} \underbrace{\left({\boldsymbol{U}^\perp}^T(\boldsymbol{x},t_k),\boldsymbol{\phi}_j(\boldsymbol{x})\right)_\Omega}_{=0} .$$

Final approximation

$$C(t_k,t_l) = \sum_{i=1}^{N_r} a_i(t_k)a_i(t_l) + \delta_{ks}\delta_{ls}\int_\Omega \sum_{i=1}^{n_c} {U^\perp}^i(\boldsymbol{x},t_s){U^\perp}^i(\boldsymbol{x},t_s)\,d\boldsymbol{x}.$$

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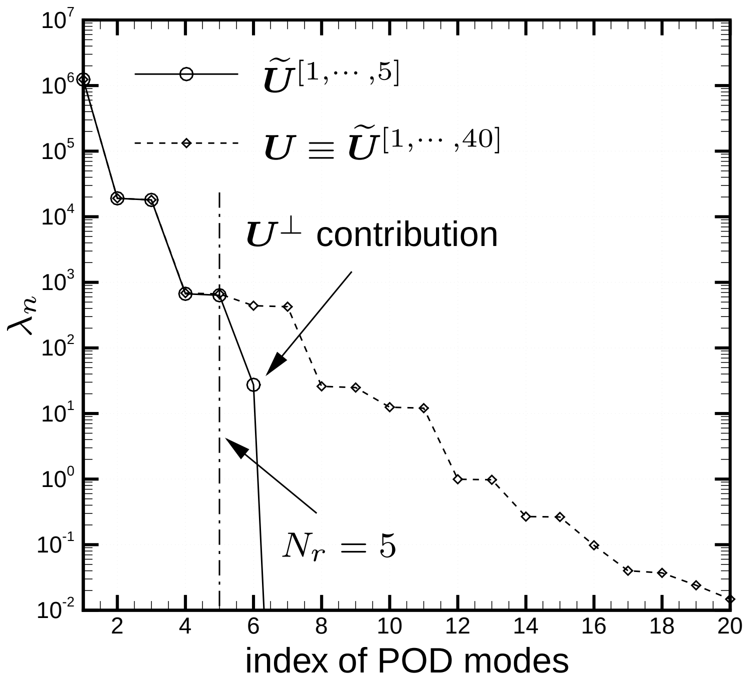
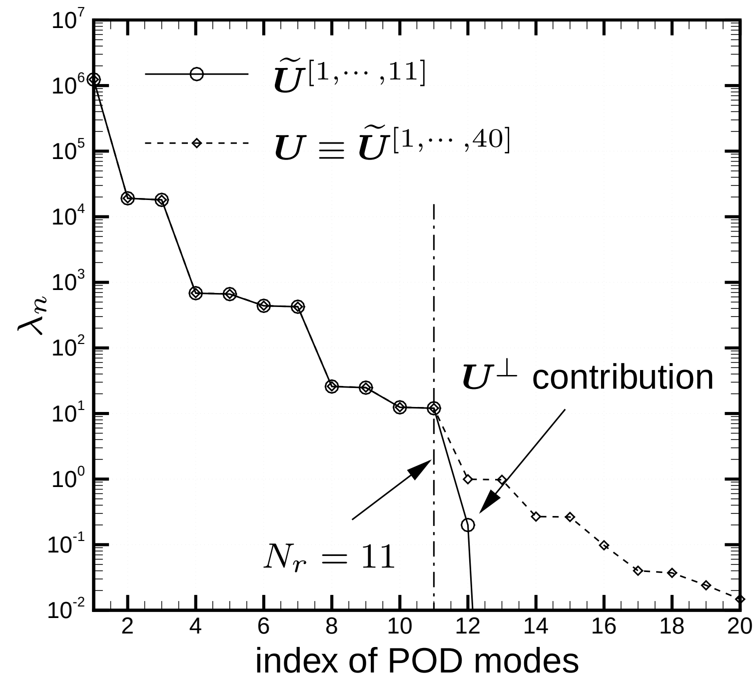
III - Improvement of the functional subspace



$N_r = 5$



$N_r = 11$

Fig. : *Comparison of the temporal correlation tensor eigenvalues evaluated from the exact field, $\boldsymbol{U}$, and from the N_r-modes approximated one, $\widetilde{\boldsymbol{U}}^{[1,\cdots,N_r]}$.*

$\hookrightarrow$ Very good approximation, and very low costs method !

III - Improvement of the functional subspace

3 Functional subspace adaptation

$$\boldsymbol{\phi}_k^{(n+1)}(\boldsymbol{x}) = \frac{1}{\lambda_k^{(n+1)}} \sum_{j=1}^{N} \widetilde{\boldsymbol{U}}^{(n)}(\boldsymbol{x}, t_j) a_k^{(n+1)}(t_j)$$

$$\boldsymbol{\phi}_k^{(n+1)}(\boldsymbol{x}) = \frac{1}{\lambda_k^{(n+1)}} \sum_{i=1}^{N_r} \sum_{j=1}^{N} a_k^{(n+1)}(t_j) a_i^{(n)}(t_j)\, \boldsymbol{\phi}_i^{(n)}(\boldsymbol{x}) + \frac{1}{\lambda_k} \boldsymbol{U}^{\perp\,(n)}(\boldsymbol{x}, t_s) a_k^{(n+1)}(t_s).$$

$$\boldsymbol{\phi}_k^{(n+1)}(\boldsymbol{x}) = \sum_{i=1}^{N_r} K_{ki}^{(n+1)} \boldsymbol{\phi}_i^{(n)}(\boldsymbol{x}) + \boldsymbol{S}_k^{(n+1)}(\boldsymbol{x}).$$

Taken $S^{(n+1)}$ with elements $S_{ij}^{(n+1)} = S_i^{j\,(n+1)}$, the actualized basis is obtained using the linear application $\varphi : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n \times \mathbb{R}^n$ defined as

$$\boxed{\varphi : \phi^{(n)} \mapsto \phi^{(n+1)} = \phi^{(n)} K^{(n+1)} + S^{(n+1)}}$$

Incrementation $n = n + 1$.

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III - Improvement of the functional subspace

▶ **Example :** we have a POD basis for $Re_1 = 100$, and we want a POD basis for $Re_2 = 200$.

- The POD ROM is evaluated with the current improved POD basis $\mathbf{\Phi}_i^{(k)}$
- The DNS is performed for Re_2.

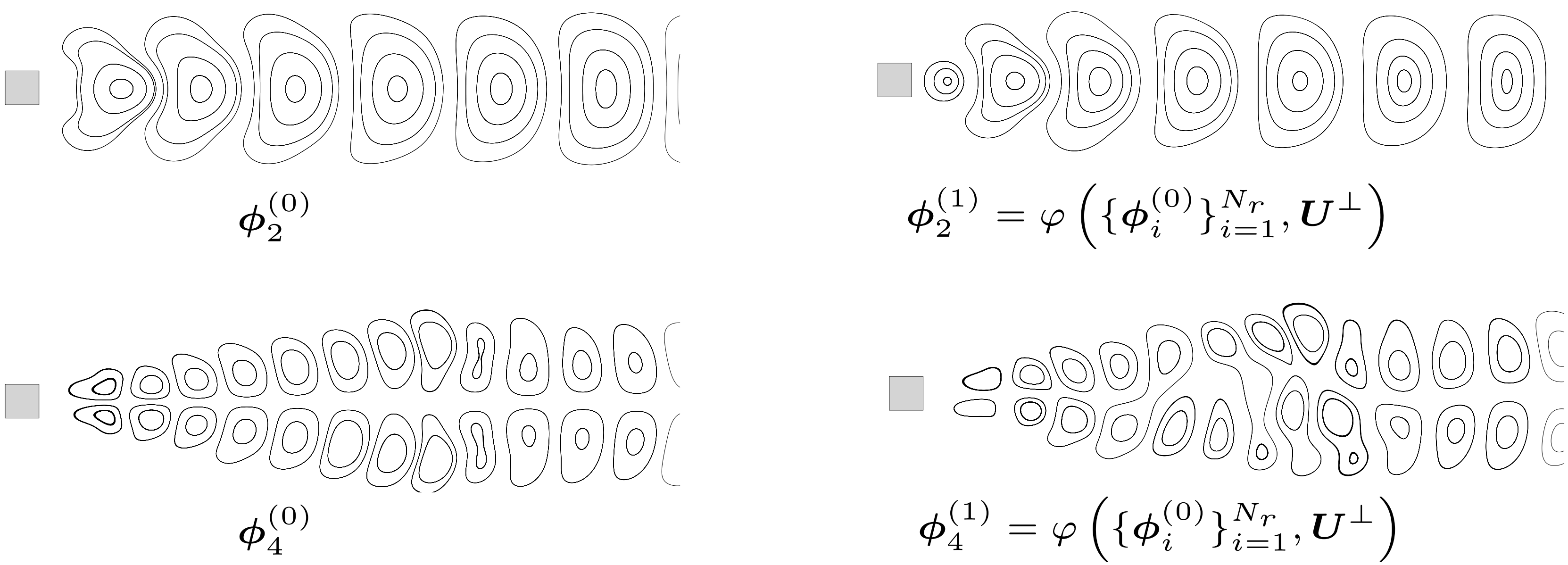


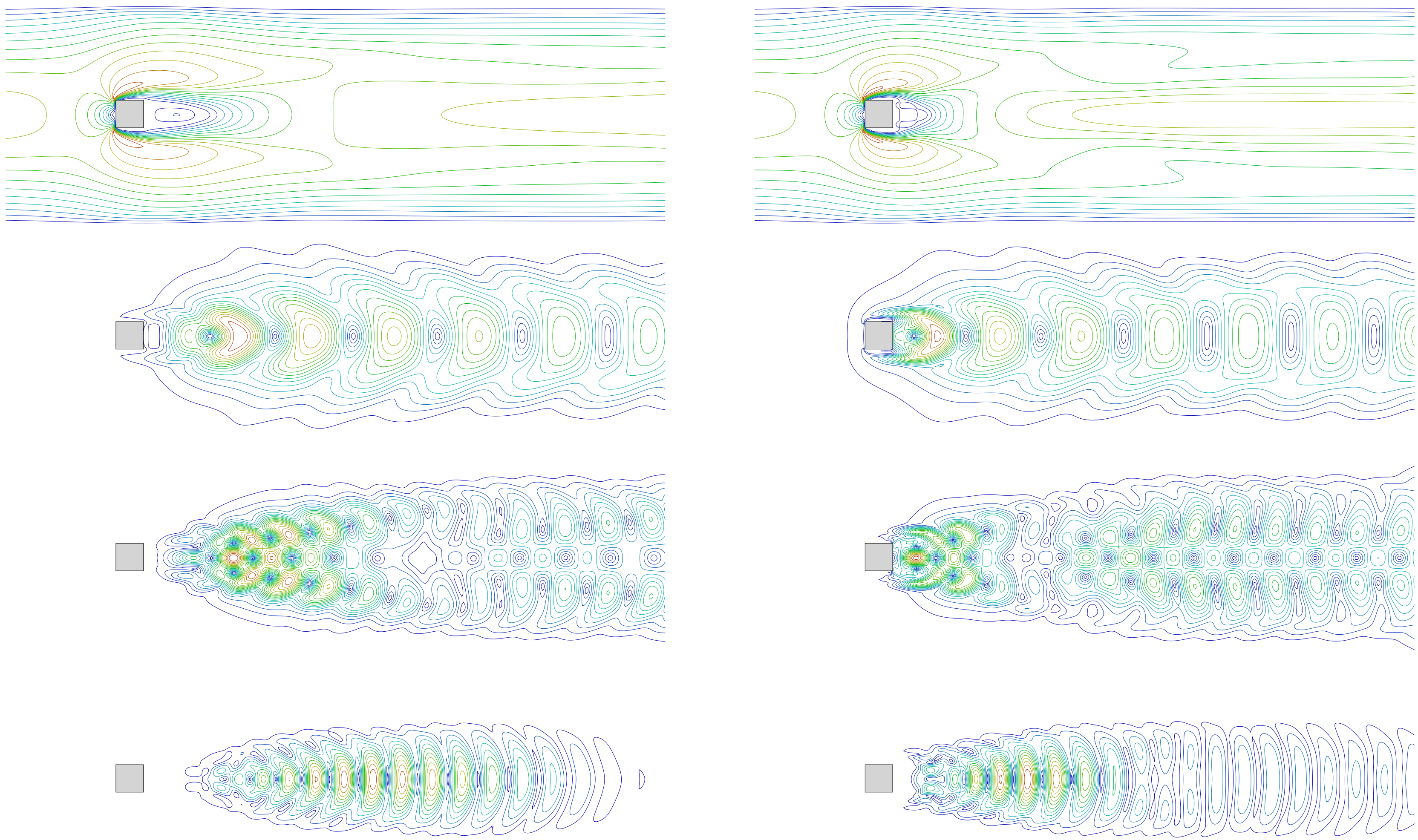
Fig. : *Modification of the POD basis functions under the application of the linear transfomation φ. Streamline representation of the velocity fields. . Bifurcation from $Re = 100$ to $Re = 200$.*

$\hookrightarrow$ Functional subspace modification

III - Improvement of the functional subspace

▶ **Results for a dynamical evolution from $Re_1 = 100$ to $Re_2 = 200$**

$Re_1 = 100$ $\qquad$ $Re_2 = 200$



III - Improvement of the functional subspace

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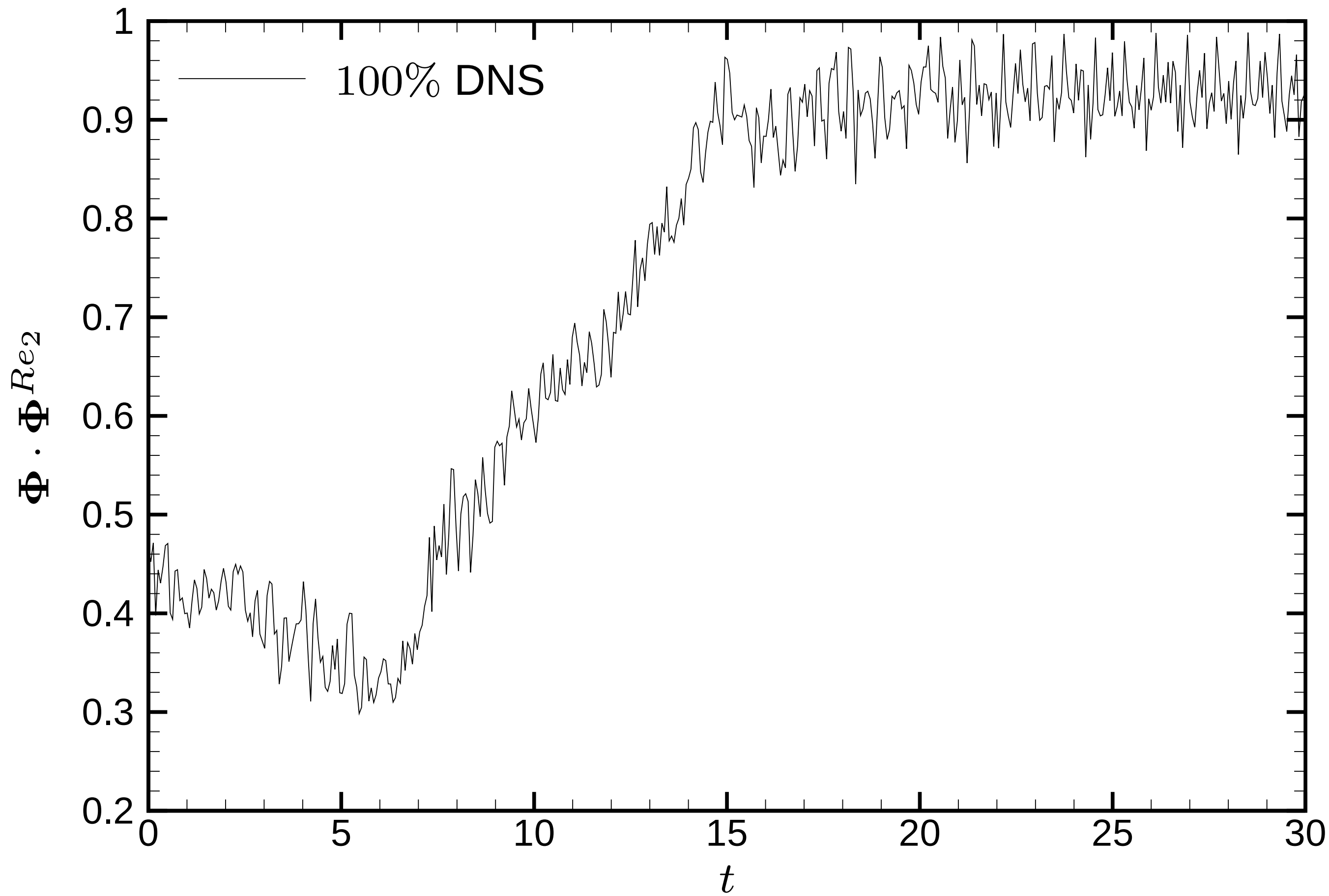

Fig. : *temporal evolution of the POD basis*

III - Improvement of the functional subspace

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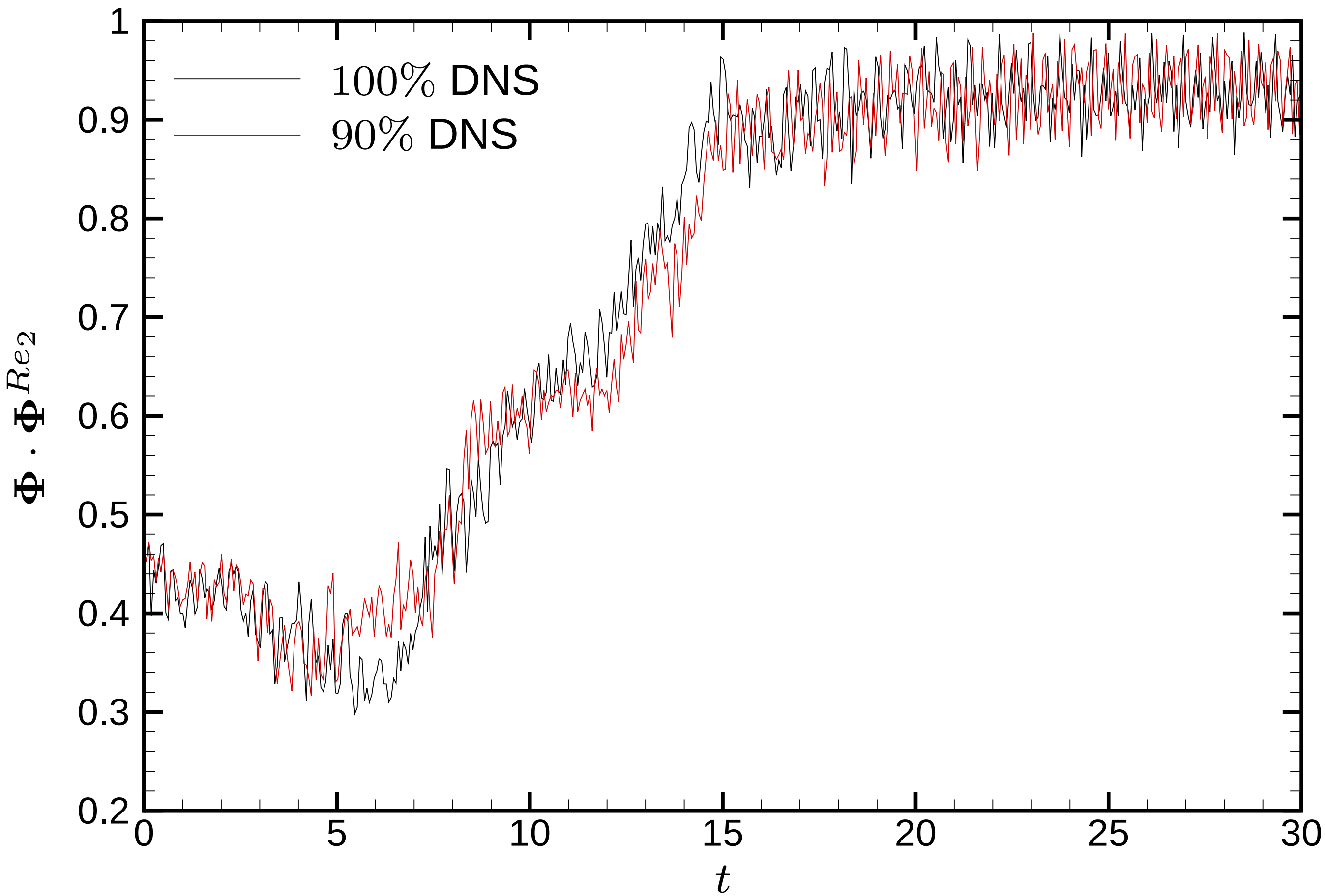

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III - Improvement of the functional subspace

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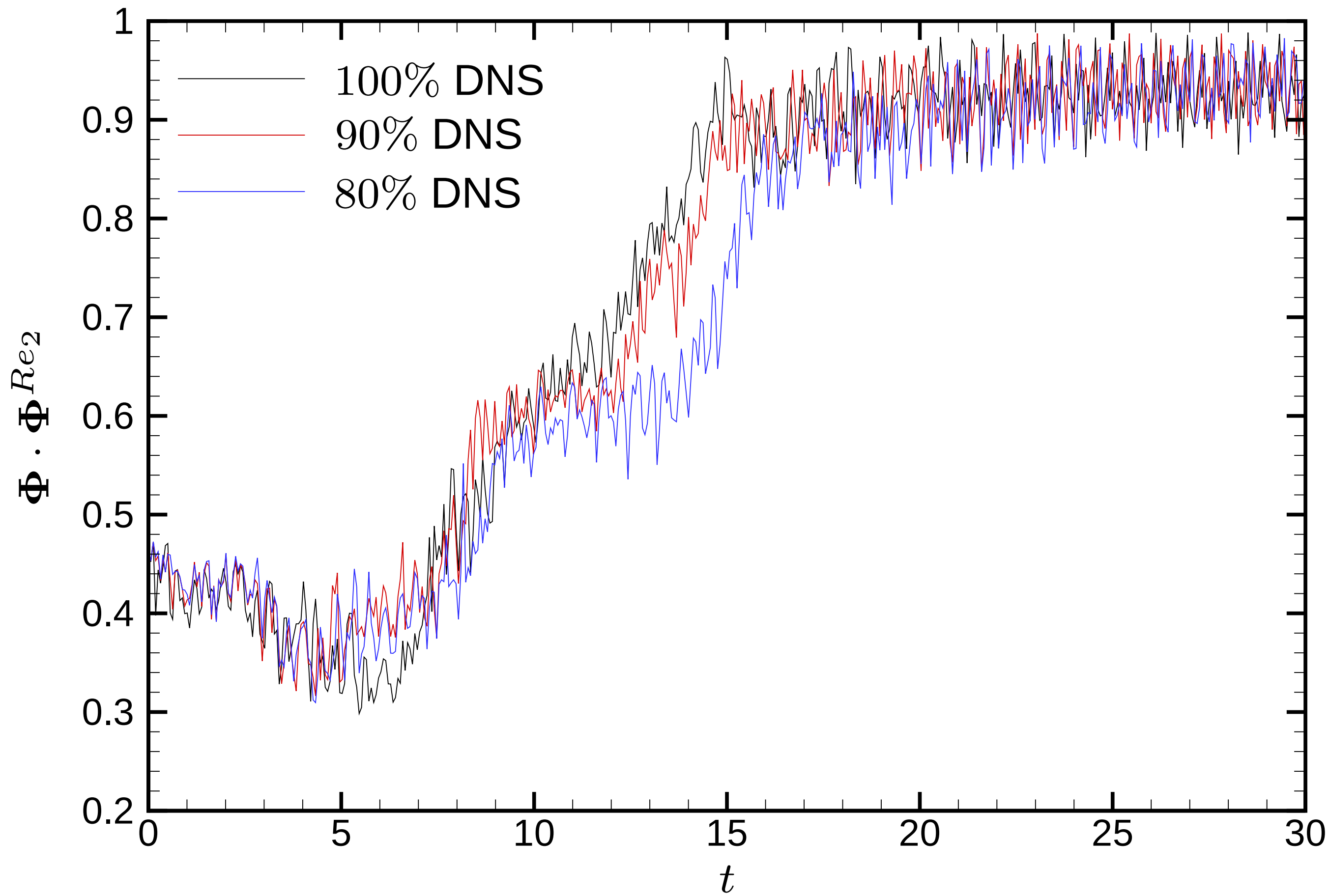

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III - Improvement of the functional subspace

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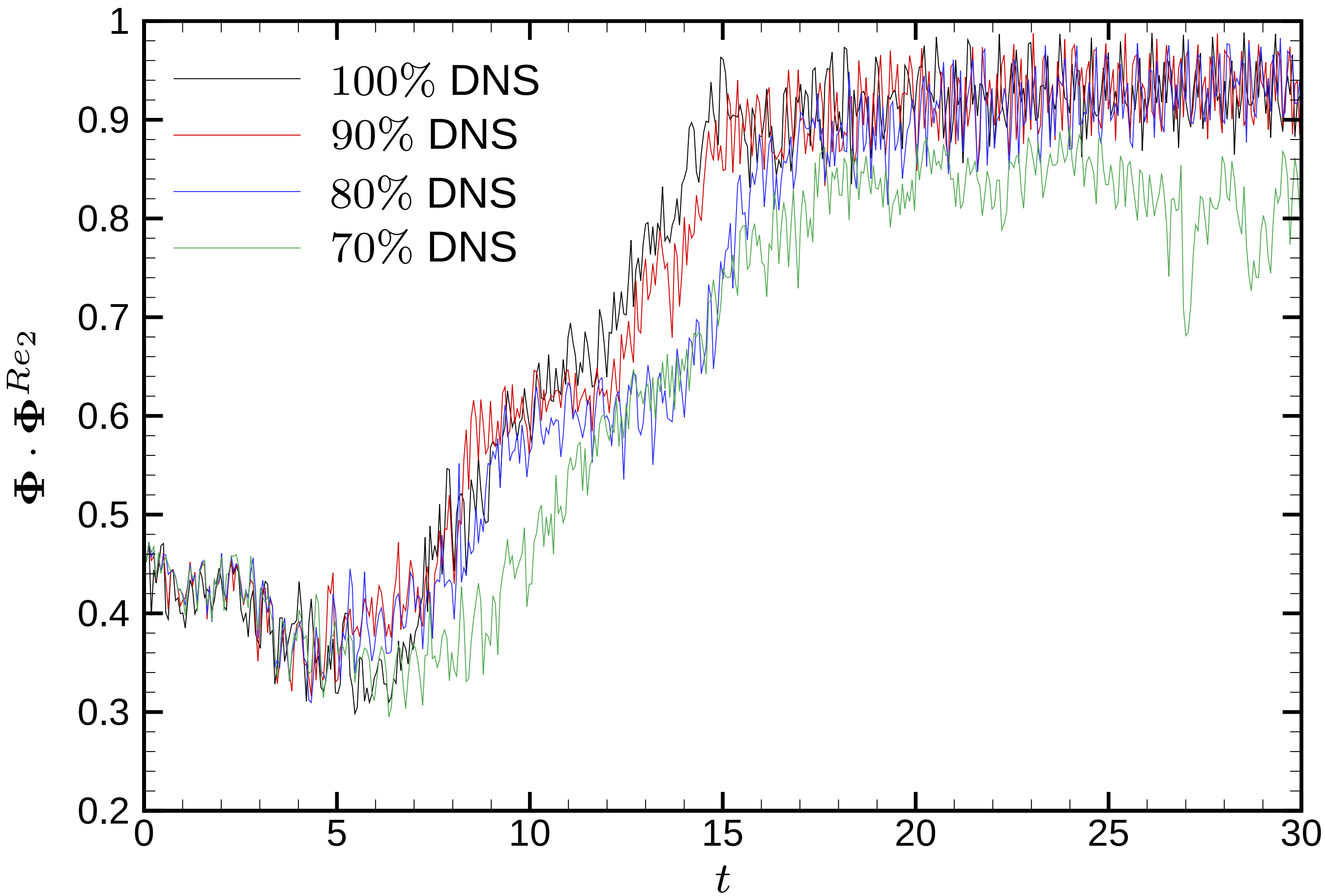

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III - Improvement of the functional subspace

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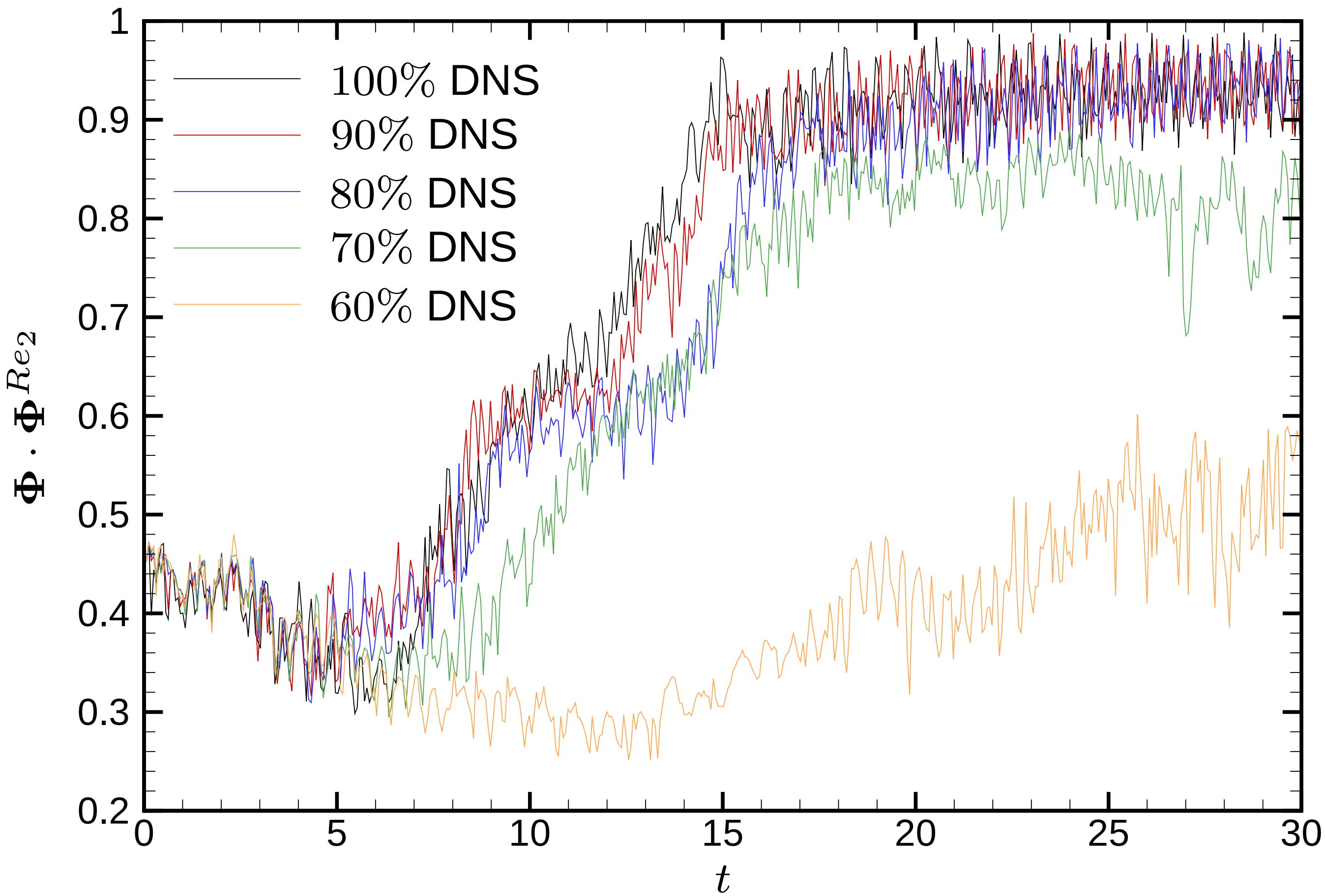

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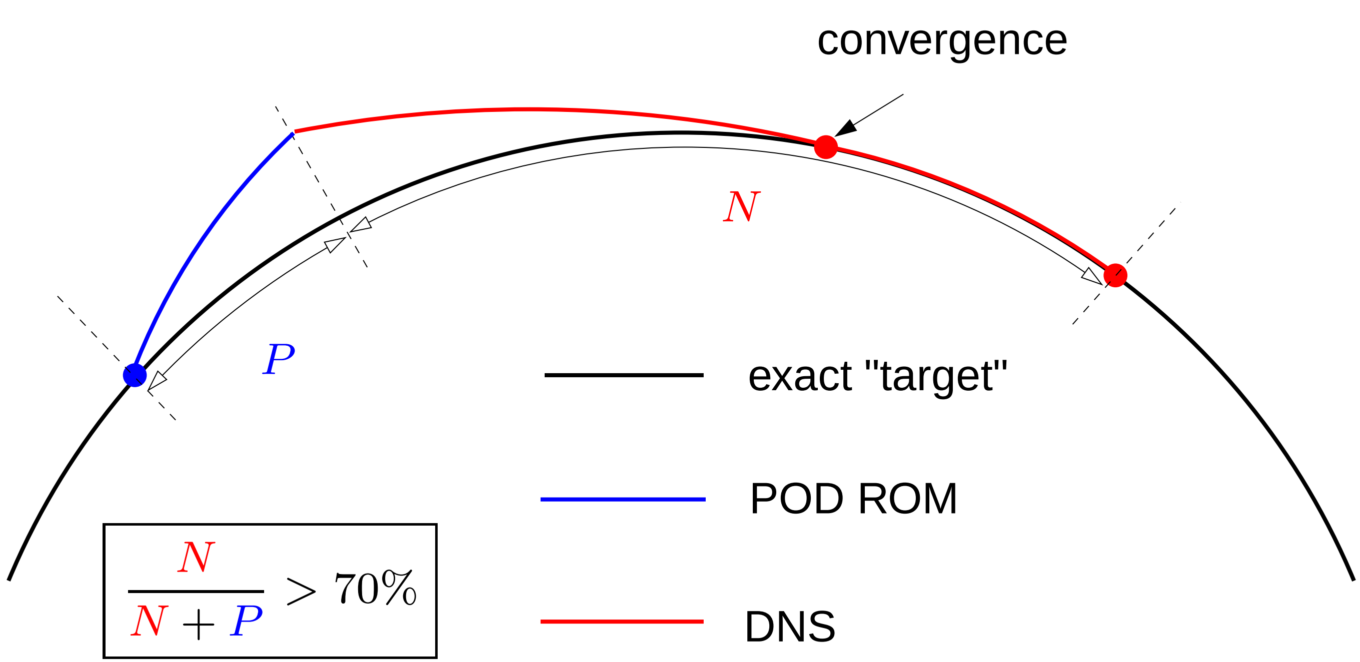
III - Improvement of the functional subspace

▶ **Observations**

- Results are very good if a sufficient amount of DNS is performed

  ↪ Good for a percentage $\frac{DNS}{DNS+PODROM}$ greater than 70%

▶ **Possible explanation**



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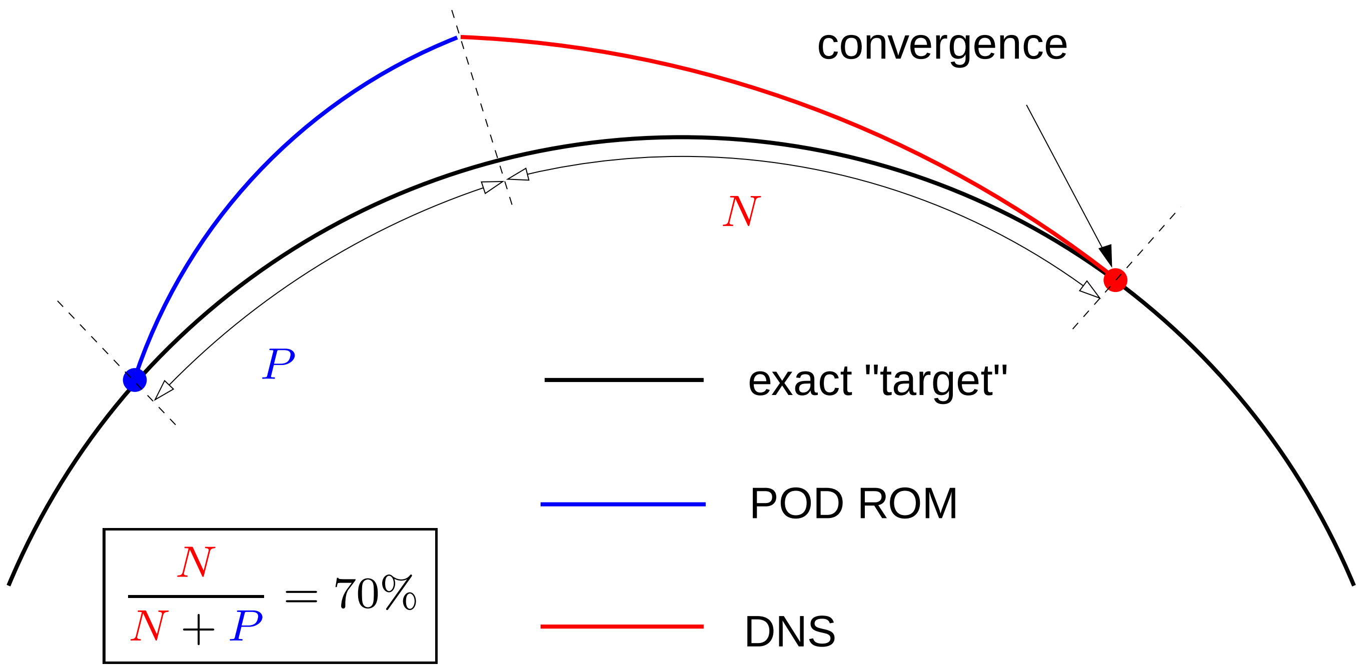

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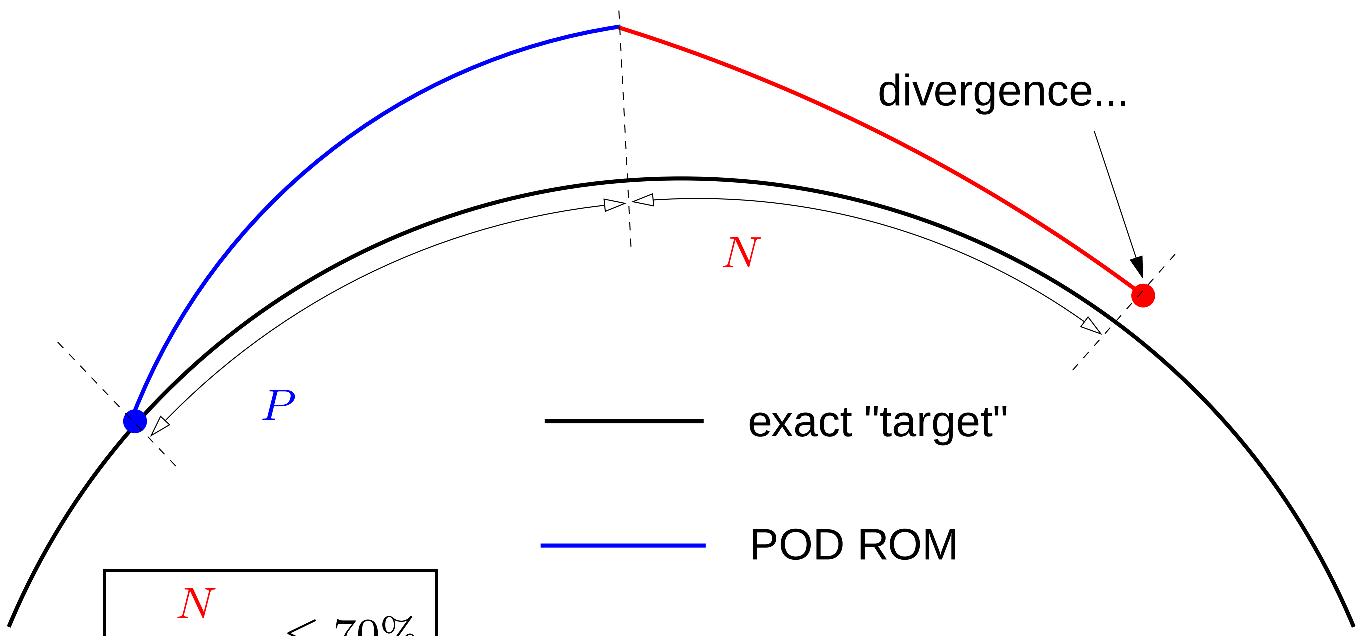

$$\frac{N}{N+P} \leq 70\%$$

III - Improvement of the functional subspace

▷ **Method 2 : Krylov-like method to improve the functional subspace $\mathbf{\Phi}_n(\boldsymbol{x})$**

∘ ***Use of the POD-NSE residuals :*** $\mathcal{L}(\widetilde{\boldsymbol{u}}(\boldsymbol{x},\, t),\, \widetilde{p}(\boldsymbol{x},\, t)) = \boldsymbol{R}(\boldsymbol{x},\, t)$,

$\widetilde{\boldsymbol{u}}$ and $\widetilde{p}$ are POD fields, $\mathcal{L}$ is the NSE operator

Algorithm

Start with the POD basis to be improved, $\mathbf{\Phi}_i$ with $i = 1, \ldots, N$. Let $N_0 = N$.

1. Built and solve the corresponding ROM to obtain $a_i(t)$ and extract N_s snapshots $a_i(t_k)$ with $i = 1, \ldots, N$ and $k = 1, \ldots, N_s$.
2. Compute $\widetilde{\boldsymbol{u}}(\boldsymbol{x},\, t_k) = \sum_{i=1}^{N} a_i(t_k)\boldsymbol{\phi}_i(\boldsymbol{x})$, $\widetilde{p}(\boldsymbol{x},\, t_k) = \sum_{i=1}^{N} a_i(t_k)\psi_i(\boldsymbol{x})$, and $\boldsymbol{R}(\boldsymbol{x},\, t_k)$.
3. Compute the POD modes $\mathbf{\Psi}(\boldsymbol{x})$ of the NSE residuals.
4. Add the K firsts residual modes $\mathbf{\Psi}(\boldsymbol{x})$ to the existing POD basis $\mathbf{\Phi}_i(\boldsymbol{x})$
   - $\mathbf{\Phi} \leftarrow \mathbf{\Phi} + \mathbf{\Psi}$
   - $N \leftarrow N + K$
   - If N is below than a threshold, return to 1. Else, go to 5.
5. Perform a new POD compression with $N = N_0$.
   - If convergence is satisfied, stop. Else, return to 1.

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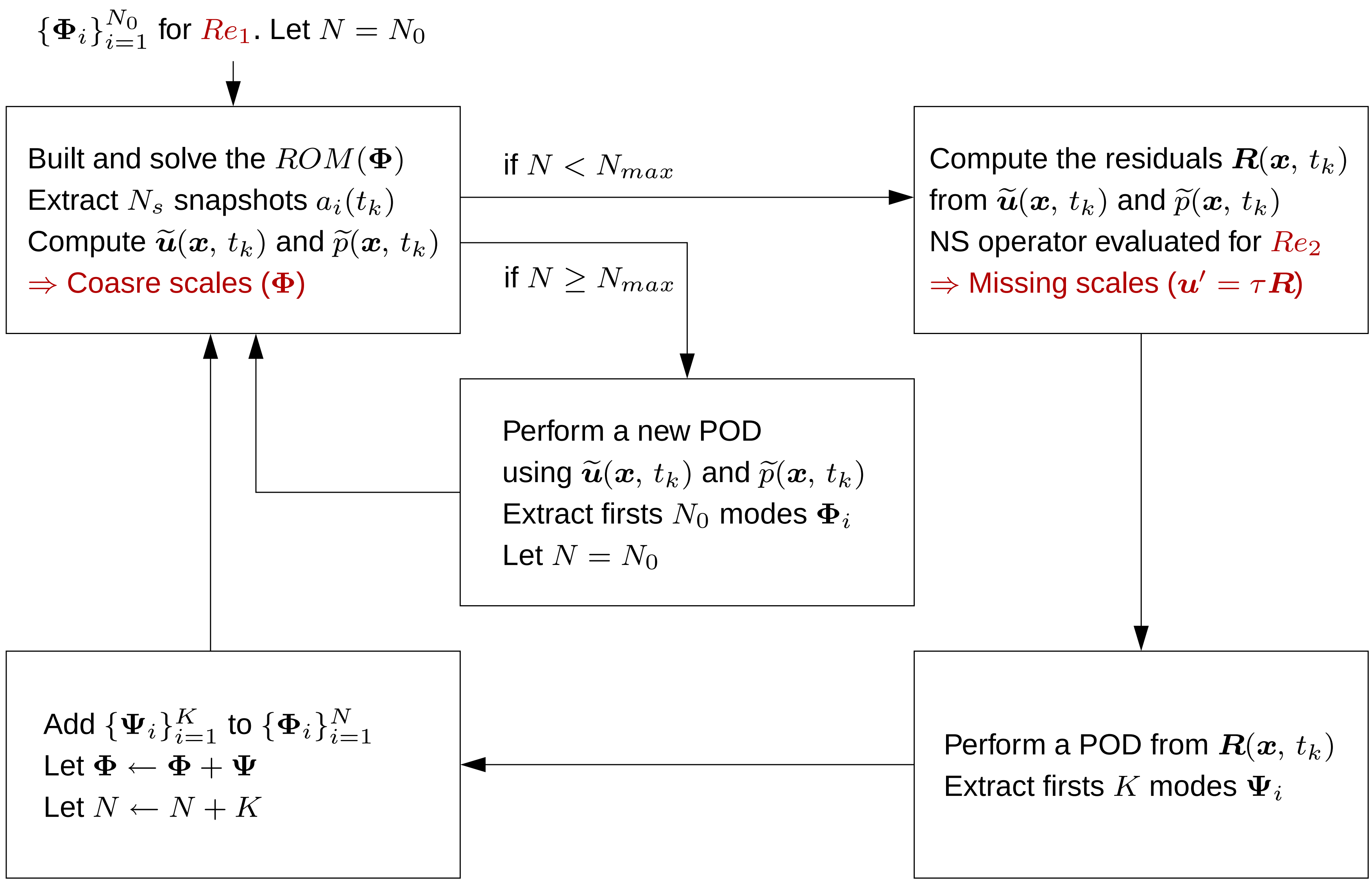
III - Improvement of the functional subspace



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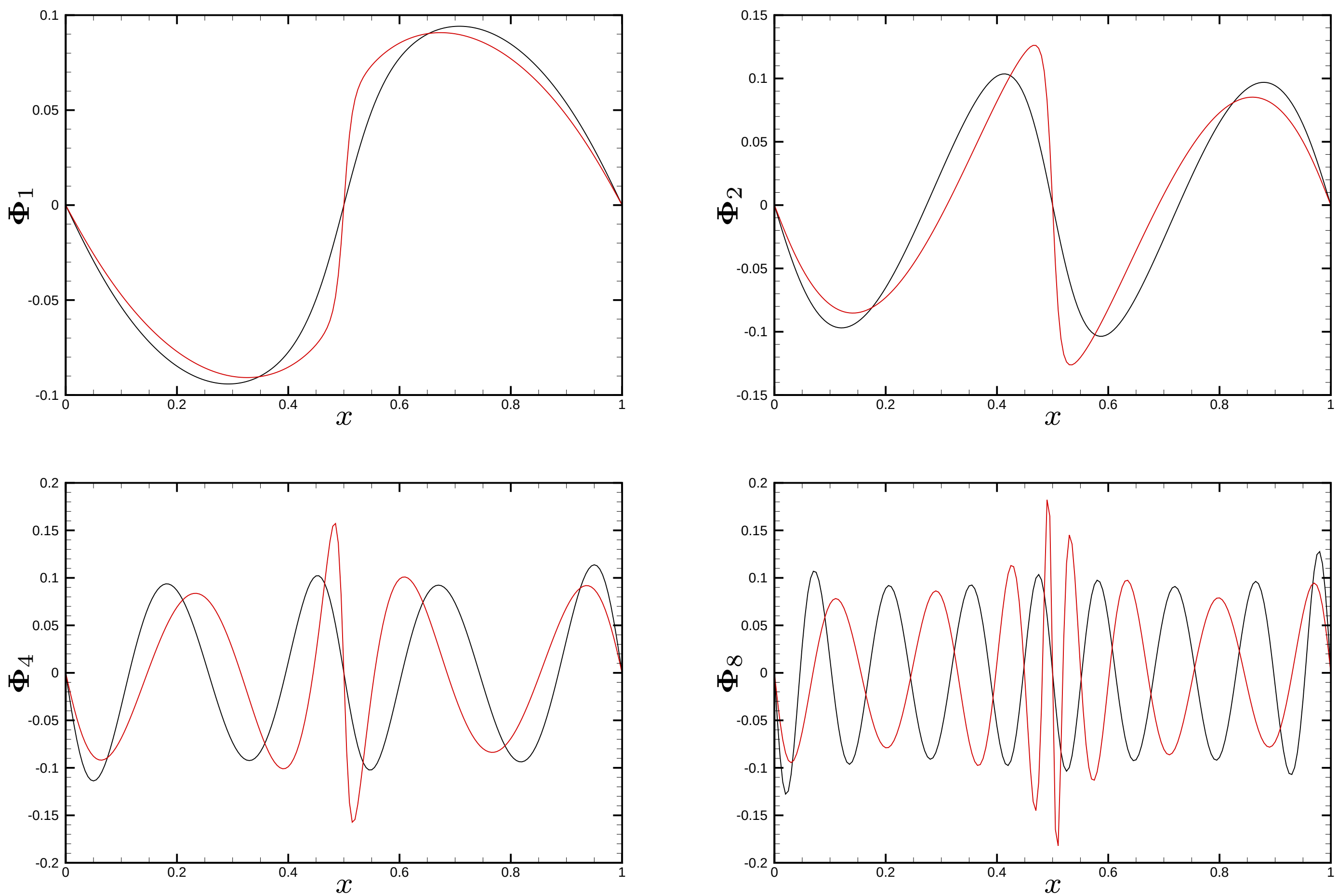
III - Improvement of the functional subspace

- **First test case :** 1D burgers equation $\underline{Re_1 = 50} \rightarrow \underline{Re_2 = 300}$ $(\Phi^{Re_1} \cdot \Phi^{Re_2} \approx 0.5)$



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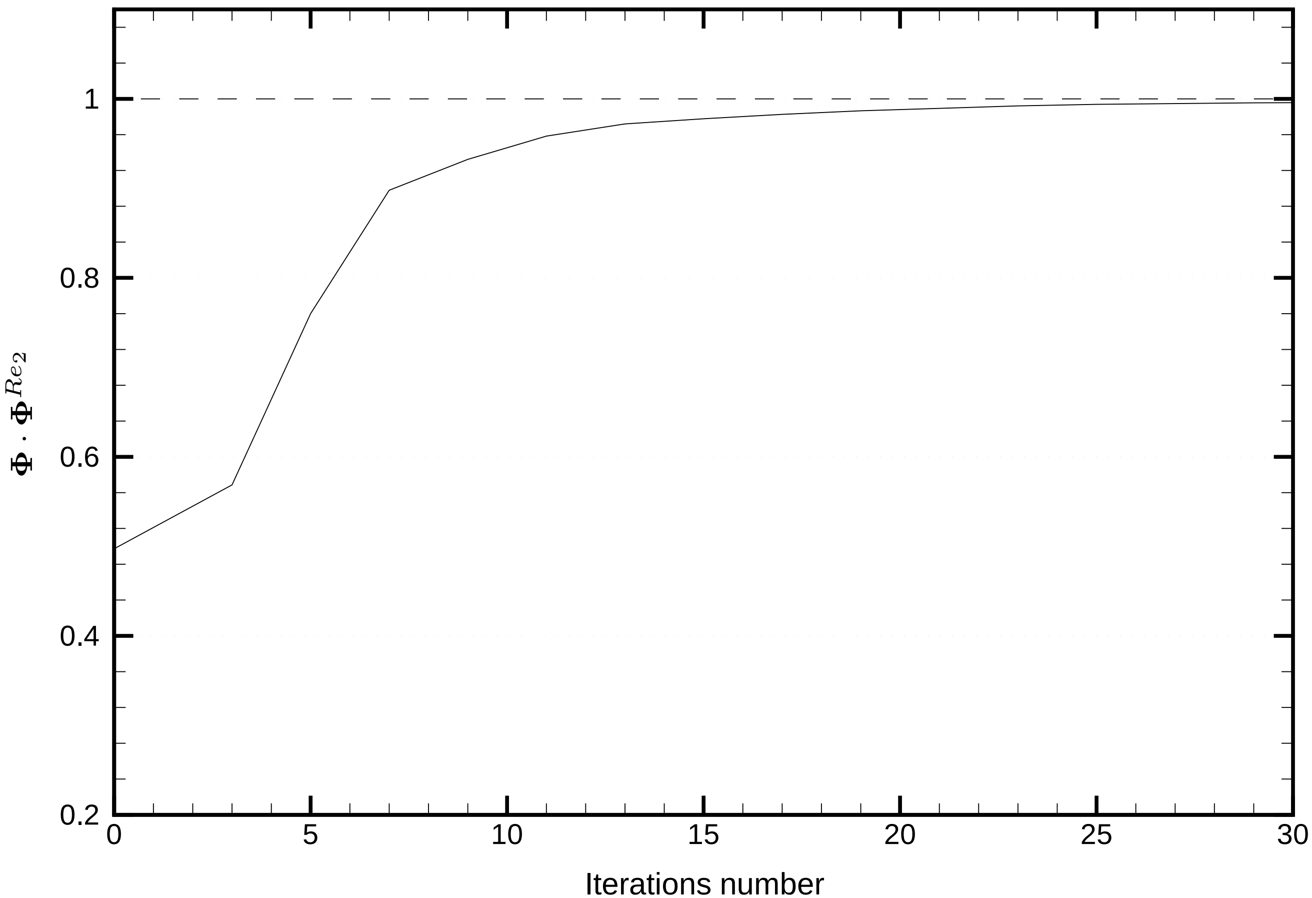

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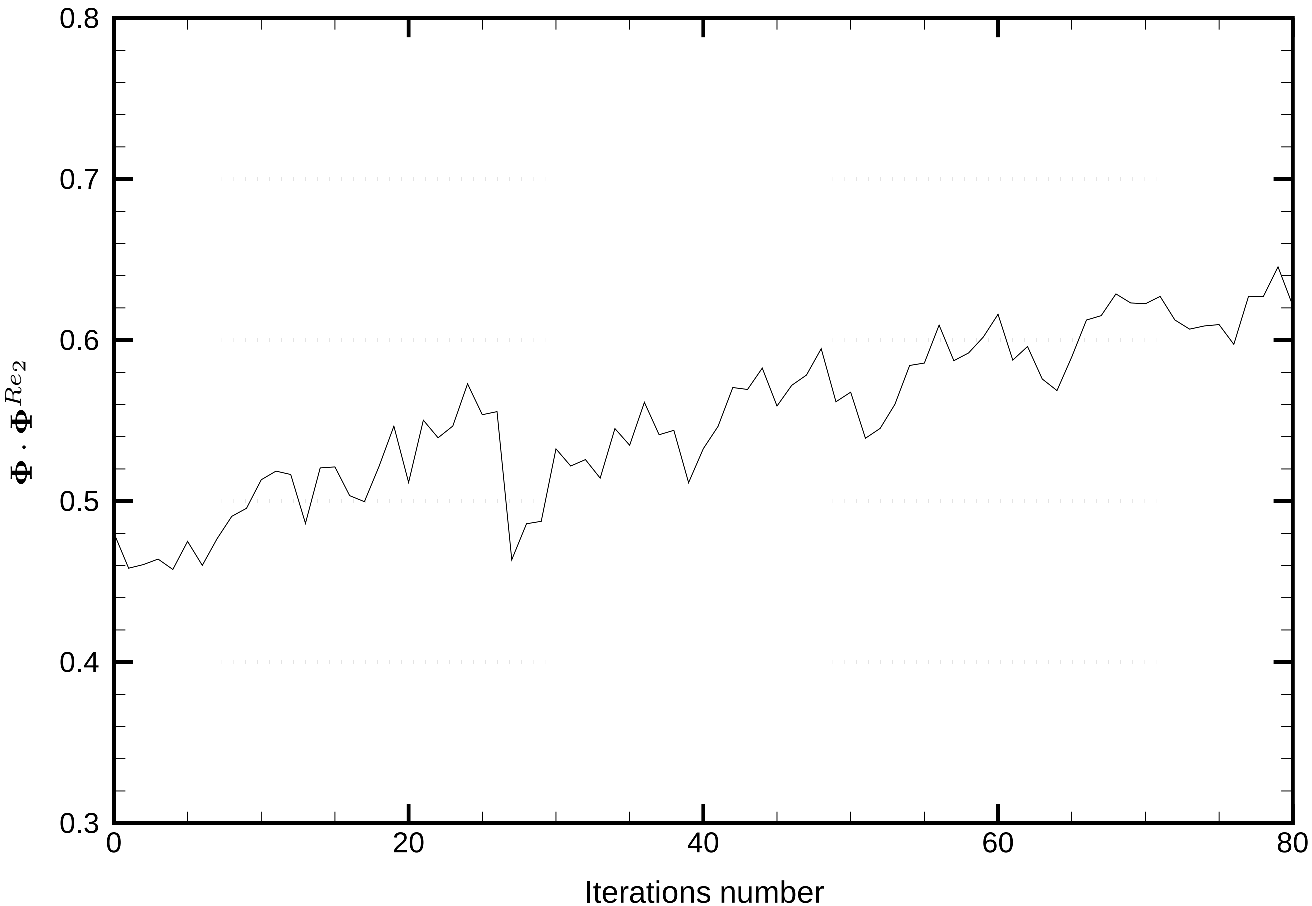
III - Improvement of the functional subspace

- **Second test case :** 2D NSE equations $\underline{Re_1 = 100} \rightarrow \underline{Re_2 = 200}$ ($\mathbf{\Phi}^{Re_1} \cdot \mathbf{\Phi}^{Re_2} \approx 0.5$)



III - Improvement of the functional subspace

▶ **Observations**

- the decomposition $\boldsymbol{U}'(\boldsymbol{x}, t) = \tau \boldsymbol{R}(\boldsymbol{x}, t)$ is used to stabilize the ROM

$\hookrightarrow$ Very good results for NSE

- the decomposition $\boldsymbol{U}'(\boldsymbol{x}, t) = \tau \boldsymbol{R}(\boldsymbol{x}, t)$ is used to improve POD basis

$\hookrightarrow$ Very good results for Burgers, quite bad results for NSE

▶ **Possible explanation**

- the decomposition $\boldsymbol{U}'(\boldsymbol{x}, t) = \tau \boldsymbol{R}(\boldsymbol{x}, t)$ is only valid for :

$\hookrightarrow$ small values of $\boldsymbol{U}'(\boldsymbol{x}, t)$ (for instance, non resolved POD modes).

$\hookrightarrow$ Can we find a good approximation τ of the elementary Green's function ? Not sure...

▶ **Future works**

- Look for an other decomposition for the missing scales $\boldsymbol{U}'(\boldsymbol{x}, t)$:

$\hookrightarrow$ $\boldsymbol{U}'(\boldsymbol{x}, t) = M(t)\boldsymbol{R}(\boldsymbol{x}, t)$, where $M \in \mathbb{R}^{3\times 3}$

$\hookrightarrow$ ... ? ?

Conclusions

- **A pressure extended Reduced Order Model**
  - The pressure is naturally included in the ROM $\Rightarrow$ no modelisation of pressure term...
  - ... but need of modelisation interaction with non resolved modes (dissipation)
- **Stabilization of Reduced Order Models based on POD**
  - Add some residual modes $\Rightarrow$ Good results
  - SUPG and VMS methods $\Rightarrow$ Very good results
- **Try to improve the functional subspace**
  - Database modification : an hybrid DNS/ROM method
    - $\hookrightarrow$ Fast evaluation of temporal correlations tensor
    - $\hookrightarrow$ Linear actualization of the POD basis
    - $\hookrightarrow$ DNS must correct ROM $\Rightarrow$ good results for amount of DNS greater than 70%
  - Improvement using POD-NSE residuals (Krylov like method)
    - $\hookrightarrow$ Very good for 1D burgers equation but quite poor results for 2D NSE equations
    - $\hookrightarrow$ Problem with the continuity equation ?
    - $\hookrightarrow$ Missing scales $\neq$ "fine scales" $\Rightarrow$ approximation $\boldsymbol{U}'(\boldsymbol{x}, t) = \tau \boldsymbol{R}(\boldsymbol{x}, t)$ not good !